日本大学歯学部紀要 31, 51–61, 2003

Behavior of the spectral function for the biharmonic operator near the boundary

Yoichi Miyazaki

Abstract

We consider the biharmonic operator with the Dirichlet boundary condition in the half space, and study the behavior of the spectral function near the boundary. We obtain the estimate for a certain integral of the spectral function, which gives a more precise remainder estimate for the counting function than the estimate for the spectral function itself.

Key words: biharmonic operator, Laplacian, spectral function, resolvent kernel, Dirichlet boundary condition

1. Introduction

As argued in the previous paper [5], it is important to study the behavior of the spectral function $e(t,x,y)$ near the boundary in order to obtain a precise remainder estimate for the counting function $N(t)$, the number of eigenvalues not exceeding t, for the elliptic operator of order $2m$ defined on a bounded domain Ω with smooth boundary in the n-dimensional Euclidean space $\mathbb{R}^n$. The spectral function is written

$$e(t,x,x) = \mu(x)t^{n/2m} + e_r(t,x,x) \tag{1.1}$$

with $\mu(x) = (2\pi)^{-n}\int_{a(x,\xi)<1} d\xi$ and the remainder term $e_r(t,x,x)$ satisfying

$$|e_r(t,x,x)| \le C\delta(x)^{-1}t^{(n-1)/2m} \qquad \text{for } t>1 \text{ and } x\in\Omega, \tag{1.2}$$

where $a(x,\xi)$ is the principal symbol of the elliptic operator and $\delta(x)$ is the distance from x to $\partial\Omega$, the boundary of Ω. This estimate was obtained independently by Brüning [2] and Tsujimoto [9]. This combined with the rough estimate

$$|e_r(t,x,x)| \le Ct^{n/2m} \qquad \text{for } t>1,$$

by Agmon [1] leads to the asymptotic formula

$$N(t) = \mu(\Omega)\,t^{n/2m} + O(t^{(n-1)/2m}\log t) \qquad \text{as } t\to\infty,$$

where $\mu(\Omega) = \int_\Omega \mu(x)\,dx$.

日本大学歯学部 数学
〒 101-8310 東京都千代田区神田駿河台 1-8-13
Department of Mathematics Nihon University School of Dentistry
1-8-13 Kanda-Surugadai, Chiyoda-ku Tokyo 101-8310, Japan
(受理：2003 年 9 月 24 日)

However, a more precise remainder estimate would be possible if we consider a certain integral of $e_r(t,x,x)$ with respect to x in place of (1.2). That is, an evaluation like

$$\left| \int_{\delta(x)<R} e_r(t,x,x)\,dx \right| \le Ct^{(n-1)/2m} \tag{1.3}$$

for some $R > 0$ could give the asymptotic formula

$$N(t) = \mu(\Omega)\, t^{n/2m} + O(t^{(n-1)/2m}) \qquad \text{as } t \to \infty.$$

In [5] the estimate (1.3) was proved for the Laplacian with the Dirichlet or the Neumann boundary condition in the half space $\mathbb{R}^n_+$. More precisely, it was shown that there exists C such that

$$\left| \int_0^R e(t,x,x)\,dx_1 \right| \le Ct^{(n-1)/2} \tag{1.4}$$

holds for any $R > 0$ and $x' \in \mathbb{R}^{n-1}$, where $x = (x_1, x') \in \mathbb{R}^n_+$. The purpose of this paper is to obtain the estimate similar to (1.4) for the biharmonic operator with the Dirichlet boundary condition.

2. Ordinary operator

In this section we consider the case of $n = 1$. Let A be the realization of D^4 with the Dirichlet boundary condition in $\mathbb{R}_+ = (0,\infty)$, or the self-adjoint operator on $L^2(\mathbb{R}_+)$ defined by

$$D(A) = H^4(\mathbb{R}_+) \cap H_0^2(\mathbb{R}_+),$$
$$Au = D^4 u = \left(\frac{1}{i}\cdot\frac{d}{dx}\right)^4 u = \frac{d^4u}{dx^4},$$

where $H^4(\mathbb{R}_+)$ and $H_0^2(\mathbb{R}_+)$ are Sobolev spaces, which will be defined clearly in the next section.

In order to find the resolvent kernel, we solve the equation $(A-\lambda)u = f$ for given $f \in L^2(\mathbb{R}_+)$ and $\lambda \in \mathbb{C} \setminus [0,\infty)$, which is equivalent to

$$\begin{cases} (D^4 - \lambda)u(x) = f(x), & x \in \mathbb{R}_+ \\ u(0) = u'(0) = 0. \end{cases}$$

Extend f to the function defined on the whole line $\mathbb{R}$ by setting $f(x) = 0$ for $x < 0$, and define v and w by

$$\begin{aligned} v(x) = (D^4-\lambda)^{-1} f(x) &= (2\pi)^{-1} \int_{\mathbb{R}} e^{ix\xi} (\xi^4 - \lambda)^{-1} \widehat{f}(\xi)\,d\xi \\ &= (2\pi)^{-1} \iint_{\mathbb{R}^2} e^{i(x-y)\xi} (\xi^4-\lambda)^{-1} f(y)\,dy\,d\xi \end{aligned} \tag{2.1}$$

and

$$u = v|_{\mathbb{R}_+} + w, \tag{2.2}$$

from which it follows that

$$\begin{cases} (D^4 - \lambda)w(x) = 0, & x \in \mathbb{R}_+ \\ w(0) = -v(0), \\ w'(0) = -v'(0). \end{cases} \tag{2.3}$$

The solutions to the equation $z^4 - \lambda = 0$ are $z = \pm\tau(\lambda), \pm\sigma(\lambda)$ with $\tau = \tau(\lambda) = e^{i\pi/4}\sqrt[4]{-\lambda}$ and $\sigma = \sigma(\lambda) = e^{3i\pi/4}\sqrt[4]{-\lambda} = i\tau(\lambda)$, where the analytic function $\sqrt[4]{z}$ is defined so that $\sqrt[4]{z} > 0$ for $z > 0$. Note that $\operatorname{Im}\tau > 0$ and $\operatorname{Im}\sigma > 0$. Since w is bounded in $\mathbb{R}_+$, the solution w to (2.3) is written

$$w(x) = ce^{i\tau(\lambda)x} + de^{i\sigma(\lambda)x},$$

with constants c and d satisfying

$$\begin{aligned} c + d = -v(0) &= -(2\pi)^{-1}\int_{\mathbb{R}} (\xi^4 - \lambda)^{-1}\widehat{f}(\xi)\, d\xi, \\ i(\tau c + \sigma d) = -v'(0) &= -(2\pi)^{-1}\int_{\mathbb{R}} i\xi(\xi^4 - \lambda)^{-1}\widehat{f}(\xi)\, d\xi. \end{aligned}$$

Consequently we have

$$w(x) = (2\pi)^{-1}\int_{\mathbb{R}} \frac{-(\xi - \sigma)e^{i\tau x} + (\xi - \tau)e^{i\sigma x}}{(\tau - \sigma)(\xi^4 - \lambda)}\widehat{f}(\xi)\, d\xi, \tag{2.4}$$

which gives the following.

Proposition 2.1. *The resolvent kernel $G_\lambda(x, y)$ for A is given by*

$$G_\lambda(x, y) = G_{0\lambda}(x, y) + G_{r\lambda}(x, y) \tag{2.5}$$

with

$$G_{0\lambda}(x, y) = (2\pi)^{-1}\int_{\mathbb{R}} e^{i(x-y)\xi}(\xi^4 - \lambda)^{-1}\, d\xi, \tag{2.6}$$

$$G_{r\lambda}(x, y) = \frac{1}{2\tau^3}\left\{\frac{-e^{i\tau(x+y)}}{2} + \frac{e^{\tau(ix-y)} + e^{\tau(-x+iy)}}{1+i} + \frac{-e^{-\tau(x+y)}}{2i}\right\}, \tag{2.7}$$

where $\tau = e^{i\pi/4}\sqrt[4]{-\lambda}$.

Proof. Note that (2.2) corresponds to (2.5). (2.6) is an immediate consequence of (2.1).

From (2.4) it follows that

$$G_{r\lambda}(x, y) = (2\pi)^{-1}\int_{\mathbb{R}} \frac{-(\xi - \sigma)e^{i\tau x} + (\xi - \tau)e^{i\sigma x}}{(\tau - \sigma)(\xi^4 - \lambda)}e^{-iy\xi}\, d\xi.$$

Changing the integral path to a lower semicircle surrounding $-\tau$ and $-\sigma$, and calculating the residues, we have

$$G_{r\lambda}(x,y) = \frac{-i}{\tau-\sigma}\left\{\frac{-e^{i\tau x+i\tau y}}{-2\tau(-\tau+\sigma)} + \frac{-e^{i\tau x+i\sigma y}}{(-\sigma-\tau)(-\sigma+\tau)} + \frac{e^{i\sigma x+i\tau y}}{(-\tau-\sigma)(-\tau+\sigma)} + \frac{e^{i\sigma x+i\sigma y}}{(-2\sigma)(-\sigma+\tau)}\right\}$$

$$= \frac{-i}{(\tau-\sigma)^2}\left\{\frac{-e^{i\tau(x+y)}}{2\tau} + \frac{e^{i\tau x+i\sigma y}+e^{i\sigma x+i\tau y}}{\tau+\sigma} + \frac{-e^{i\sigma(x+y)}}{2\sigma}\right\},$$

which gives (2.7). □

Proposition 2.2. *The spectral function $e(t,x,y)$ for A is given by*

$$e(t,x,y) = e_0(t,x-y) + e_r(t,x,y),$$

where

$$e_0(t,z) = (2\pi)^{-1}\int_{\xi^4\le t} e^{iz\xi}\,d\xi \tag{2.8}$$

and $e_r(t,x,y)$ satisfies

$$|e_r(t,x,x)| \le \frac{3}{\pi x} \tag{2.9}$$

for $t>0$ and $x>0$.

Proof. The spectral function is obtained from the resolvent kernel by the formula

$$e(t,x,y) = \frac{1}{2\pi i}\lim_{\delta\to+0}\lim_{\varepsilon\to+0}\int_{\Gamma(t,\delta,\varepsilon)} G_\lambda(x,y)\,d\lambda, \tag{2.10}$$

where $\Gamma(t,\delta,\varepsilon)$ is a curve in the complex plane from $t+\delta-i\varepsilon$ to $t+\delta+i\varepsilon$ not intersecting $[0,\infty)$. $e_0(t,x-y)$ and $e_r(t,x,y)$ correspond to $G_{0\lambda}(x,y)$ and $G_{r\lambda}(x,y)$ respectively. The formula (2.8) for the principal term $e_0(t,x-y)$ is an immediate consequence of (2.6).

Applying (2.10) and using

$$\tau(s+i0) = s^{1/4}, \qquad \tau(s-i0) = is^{1/4},$$

we have

$$e_r(t,x,y) = \frac{1}{2\pi i}\lim_{\delta\to+0}\int_0^{t+\delta}\{G_{r(s+i0)}(x,y) - G_{r(s-i0)}(x,y)\}\,ds$$

$$= \int_0^t \frac{1}{4\pi i s^{3/4}}\left\{\frac{-e^{is^{1/4}(x+y)}}{2} + \frac{e^{s^{1/4}(ix-y)}+e^{s^{1/4}(-x+iy)}}{1+i} + \frac{-e^{-s^{1/4}(x+y)}}{2i}\right\}ds$$

$$- \int_0^t \frac{1}{4\pi s^{3/4}}\left\{\frac{-e^{-s^{1/4}(x+y)}}{2} + \frac{e^{is^{1/4}(ix-y)}+e^{is^{1/4}(-x+iy)}}{1+i} + \frac{-e^{-is^{1/4}(x+y)}}{2i}\right\}ds$$

$$= \int_0^{t^{1/4}} \frac{1}{\pi}\left\{e^{-s(x+y)} - \sin s(x+y) + 2\mathrm{Re}\,\frac{e^{s(-x+iy)}+e^{s(ix-y)}}{-1+i}\right\}ds.$$

By setting $y = x$ we get

$$e_r(t,x,x) = \int_0^{t^{1/4}} \frac{1}{\pi}\left\{e^{-2sx} - \sin 2sx + 4\mathrm{Re}\,\frac{e^{(i-1)sx}}{-1+i}\right\} ds \tag{2.11}$$
$$= \frac{1}{2\pi x}\{-e^{-2t^{1/4}x} + \cos(2t^{1/4}x) - 4e^{-t^{1/4}x}\sin(t^{1/4}x)\},$$

from which the proposition follows. □

Proposition 2.3. *There exists a constant $C > 0$ such that*

$$\left|\int_0^R e_r(t,x,x)\,dx\right| \le C$$

for any $R > 0$ and $t > 0$.

Proof. Integrating (2.11), we have

$$\int_0^R e_r(t,x,x)\,dx = \int_0^{t^{1/4}} \frac{1}{\pi}\left\{\frac{\cos 2sR - e^{-2sR}}{2s} + 4\mathrm{Re}\,\frac{e^{(i-1)sR}-1}{-2is}\right\} ds$$
$$= \int_0^{Rt^{1/4}} \frac{1}{2\pi s}\left\{\cos 2s - e^{-2s} - 4e^{-s}\sin s\right\} ds.$$

It is easily seen that $\int_0^1 s^{-1}\left|\cos 2s - e^{-2s}\right| < \infty$, $\sup_{R>1}\int_1^R (\cos 2s)/s\,ds < \infty$ and $|\sin s| \le s$ for $s > 0$. Therefore the proposition follows. □

3. The case of $n > 1$

We can treat the case of $n > 1$ in the same way as the case of $n = 1$, if we use the partial Fourier transform. Let $\mathbb{R}^n_+$ be the half space of $\mathbb{R}^n$, that is, the set of points $x = (x_1, x') = (x_1, x_2, \ldots, x_n)$ with $x_1 > 0$. The conjugate variable of x is denoted by $\xi = (\xi_1, \xi') = (\xi_1, \xi_2, \ldots, \xi_n)$. Let A be the realization of the biharmonic operator Δ^2 with the Dirichlet boundary condition in $\mathbb{R}^n_+$, or the self-adjoint operator on $L^2(\mathbb{R}^n_+)$ defined by

$$D(A) = H^4(\mathbb{R}^n_+) \cap H^2_0(\mathbb{R}^n_+),$$
$$Au = \Delta^2 u = \left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right)^2 u,$$

where $H^k(\mathbb{R}^n_+)$ denotes the Sobolev space of order k and $H^k_0(\mathbb{R}^n_+)$ is the closure of $C_0^\infty(\mathbb{R}^n_+)$, the set of infinitely differentiable functions with compact support, in $H^k(\mathbb{R}^n_+)$.

In order to find the resolvent kernel, we solve the equation $(A-\lambda)u = f$ for given $f \in L^2(\mathbb{R}^n_+)$ and $\lambda \in \mathbb{C} \setminus [0,\infty)$, which is equivalent to

$$\begin{cases} \left((D_1^2 + \cdots + D_n^2)^2 - \lambda\right) u(x) = f(x), & x \in \mathbb{R}^n_+ \\ u(0,x') = 0, & x' \in \mathbb{R}^{n-1} \\ \partial_1 u(0,x') = 0, & x' \in \mathbb{R}^{n-1}. \end{cases}$$

Extend f to the function defined on the whole space $\mathbb{R}^n$ by defining $f(x) = 0$ for $x_1 < 0$, and define v and w by

$$v = (\Delta^2 - \lambda)^{-1} f = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\xi} (|\xi|^4 - \lambda)^{-1} \widehat{f}(\xi)\, d\xi \tag{3.1}$$
$$= (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} e^{i(x-y)\xi} (|\xi|^4 - \lambda)^{-1} f(y)\, dy\, d\xi$$

and

$$u = v|_{\mathbb{R}^n_+} + w, \tag{3.2}$$

from which it follows that

$$\begin{cases} \left((D_1^2 + \cdots + D_n^2)^2 - \lambda\right) w(x) = 0, & x \in \mathbb{R}^n_+ \\ w(0, x') = -v(0, x'), & x' \in \mathbb{R}^{n-1} \\ \partial_1 w(0, x') = -\partial_1 v(0, x'), & x' \in \mathbb{R}^{n-1}. \end{cases} \tag{3.3}$$

To solve (3.3) we define the partial Fourier transform with respect to x' by

$$\mathcal{F}' w(x_1, \xi') = \int_{\mathbb{R}^{n-1}} e^{-ix'\xi'} w(x_1, x')\, dx'$$

and its inverse

$$\mathcal{F}'^{-1} f(x_1, x') = (2\pi)^{1-n} \int_{\mathbb{R}^{n-1}} e^{ix'\xi'} f(x_1, \xi')\, d\xi'.$$

Applying $\mathcal{F}'$ to (3.3), we have

$$\begin{cases} \left(\left(-\frac{d^2}{dx_1^2} + |\xi'|^2\right)^2 - \lambda\right) \mathcal{F}' w(x_1, \xi') = 0 \\ \mathcal{F}' w(0, \xi') = -\mathcal{F}' v(0, \xi') \\ \partial_1 \mathcal{F}' w(0, \xi') = -\partial_1 \mathcal{F}' v(0, \xi'). \end{cases} \tag{3.4}$$

Fourier's inversion formula and (3.1) give

$$\begin{aligned} \mathcal{F}' v(0, \xi') &= \int_{\mathbb{R}^{n-1}} e^{-ix'\xi'} v(0, x')\, dx' \\ &= (2\pi)^{-n} \int_{\mathbb{R}^{n-1}} e^{-ix'\xi'}\, dx' \int_{\mathbb{R}^n} e^{ix'\eta'} ((\xi_1^2 + |\eta'|^2)^2 - \lambda)^{-1} \widehat{f}(\xi_1, \eta')\, d\xi_1\, d\eta' \\ &= (2\pi)^{-1} \int_{\mathbb{R}} (|\xi|^4 - \lambda)^{-1} \widehat{f}(\xi)\, d\xi_1, \end{aligned}$$

$$\begin{aligned} \partial_1 \mathcal{F}' v(0, \xi') &= \int_{\mathbb{R}^{n-1}} e^{-ix'\xi'} \partial_1 v(0, x')\, dx' \\ &= (2\pi)^{-n} \int_{\mathbb{R}^{n-1}} e^{-ix'\xi'}\, dx' \int_{\mathbb{R}^n} e^{ix'\eta'} i\xi_1 ((\xi_1^2 + |\eta'|^2)^2 - \lambda)^{-1} \widehat{f}(\xi_1, \eta')\, d\xi_1\, d\eta' \\ &= (2\pi)^{-1} \int_{\mathbb{R}} i\xi_1 (|\xi|^4 - \lambda)^{-1} \widehat{f}(\xi)\, d\xi_1. \end{aligned}$$

The solutions to the equation $(z^2+|\xi'|^2)^2-\lambda=0$ are $z=\pm\tau(\lambda,\xi'),\ \pm\sigma(\lambda,\xi')$ with $\tau=\tau(\lambda,\xi')=\sqrt{-|\xi'|^2+i\sqrt{-\lambda}}$ and $\sigma=\sigma(\lambda,\xi')=-\sqrt{-|\xi'|^2-i\sqrt{-\lambda}}$, where the analytic function $\sqrt{z}$ is defined so that $\sqrt{z}>0$ for $z>0$. Note that $\operatorname{Im}\tau>0$ and $\operatorname{Im}\sigma>0$. Since $\mathcal{F}'w(x_1,\xi')$ are bounded in $\mathbb{R}^n_+$, the solution $\mathcal{F}'w(x_1,\xi')$ to (3.4) is written

$$\mathcal{F}'w(x_1,\xi')=c(\lambda,\xi')e^{i\tau(\lambda,\xi')x_1}+d(\lambda,\xi')e^{i\sigma(\lambda,\xi')x_1},$$

where

$$c(\lambda,\xi')+d(\lambda,\xi')=-\mathcal{F}'v(0,\xi')=-(2\pi)^{-1}\int_{\mathbb{R}}(|\xi|^4-\lambda)^{-1}\widehat{f}(\xi)\,d\xi_1,$$

$$i\tau c(\lambda,\xi')+i\sigma d(\lambda,\xi')=-\partial_1\mathcal{F}'v(0,\xi')=-(2\pi)^{-1}\int_{\mathbb{R}}i\xi_1(|\xi|^4-\lambda)^{-1}\widehat{f}(\xi)\,d\xi_1.$$

Consequently we have

$$\mathcal{F}'w(x_1,\xi')=(2\pi)^{-1}\int_{\mathbb{R}}\frac{-(\xi_1-\sigma)e^{i\tau x_1}+(\xi_1-\tau)e^{i\sigma x_1}}{(\tau-\sigma)((\xi_1^2+|\xi'|^2)^2-\lambda)}\widehat{f}(\xi)\,d\xi_1,$$

and therefore

$$w(x)=(2\pi)^{-n}\iiint_{\mathbb{R}^{2n}}\frac{-(\xi_1-\sigma)e^{i\tau x_1}+(\xi_1-\tau)e^{i\sigma x_1}}{(\tau-\sigma)((\xi_1^2+|\xi'|^2)^2-\lambda)}e^{-iy_1\xi_1+i(x'-y')\xi'}f(y)\,dy\,d\xi_1\,d\xi',\quad (3.5)$$

which gives the following.

Proposition 3.1. *The resolvent kernel $G_\lambda(x,y)$ for A is given by*

$$G_\lambda(x,y)=G_{0\lambda}(x,y)+G_{r\lambda}(x,y)\quad (3.6)$$

with

$$G_{0\lambda}(x,y)=(2\pi)^{-n}\int_{\mathbb{R}^n}e^{i(x-y)\xi}(|\xi|^4-\lambda)^{-1}\,d\xi,\quad (3.7)$$

$$G_{r\lambda}(x,y)=\int_{\mathbb{R}^{n-1}}\frac{i(2\pi)^{1-n}}{(\tau-\sigma)^2}\left\{\frac{e^{i\tau(x_1+y_1)}}{2\tau}-\frac{e^{i\sigma x_1+i\tau y_1}+e^{i\tau x_1+i\sigma y_1}}{\tau+\sigma}+\frac{e^{i\sigma(x_1+y_1)}}{2\sigma}\right\}e^{i(x'-y')\xi'}\,d\xi',\quad (3.8)$$

where $\tau=\sqrt{-|\xi'|^2+i\sqrt{-\lambda}}$ and $\sigma=-\sqrt{-|\xi'|^2-i\sqrt{-\lambda}}$.

Proof. Note that (3.2) corresponds to (3.6). (3.7) is an immediate consequence of (3.1).

Using (3.5) and the residue theorem, we have

$$\begin{aligned}G_{r\lambda}(x,y)&=(2\pi)^{-n}\iint_{\mathbb{R}^n}\frac{-(\xi_1-\sigma)e^{i\tau x_1}+(\xi_1-\tau)e^{i\sigma x_1}}{(\tau-\sigma)((\xi_1^2+|\xi'|^2)^2-\lambda)}e^{-iy_1\xi_1+i(x'-y')\xi'}\,d\xi_1\,d\xi'\\&=\int_{\mathbb{R}^{n-1}}\frac{-i(2\pi)^{1-n}}{\tau-\sigma}\left\{\frac{-e^{i\tau x_1+i\tau y_1}}{-2\tau(-\tau+\sigma)}+\frac{e^{i\sigma x_1+i\tau y_1}}{(-\tau-\sigma)(-\tau+\sigma)}\right.\\&\qquad\left.+\frac{-e^{i\tau x_1+i\sigma y_1}}{(-\sigma-\tau)(-\sigma+\tau)}+\frac{e^{i\sigma x_1+i\sigma y_1}}{(-2\sigma)(-\sigma+\tau)}\right\}e^{i(x'-y')\xi'}\,d\xi',\end{aligned}$$

from which (3.8) follows. □

Proposition 3.2. *The spectral function $e(t,x,y)$ for A is given by*

$$e(t,x,y) = e_0(t,x-y) + e_r(t,x,y),$$

where

$$e_0(t,z) = (2\pi)^{-n} \int_{|\xi|^4 \le t} e^{iz\xi}\, d\xi \tag{3.9}$$

and $e_r(t,x,x)$ satisfies

$$|e_r(t,x,x)| \le C x_1^{-1} t^{(n-1)/4} \tag{3.10}$$

for any $t > 0$ and $x = (x_1, x') \in \mathbb{R}^n_+$ with some $C > 0$.

Proof. The spectral function is obtained from the resolvent kernel by (2.10). The formula (3.9) for the principal term $e(t,x-y)$ follows from (3.7).

Applying (2.10) with (3.8) and using

$$\tau(s+i0,\xi') = \alpha(s,\xi'), \qquad \sigma(s+i0,\xi') = i\beta(s,\xi')$$
$$\tau(s-i0,\xi') = i\beta(s,\xi'), \qquad \sigma(s-i0,\xi') = -\alpha(s,\xi')$$

for $|\xi'|^4 \le s$, and

$$\tau(s+i0,\xi') = i\gamma(s,\xi'), \qquad \sigma(s+i0,\xi') = i\beta(s,\xi')$$
$$\tau(s-i0,\xi') = i\beta(s,\xi'), \qquad \sigma(s-i0,\xi') = i\gamma(s,\xi')$$

for $0 \le s \le |\xi'|^4$, where

$$\alpha(s,\xi') = \sqrt{\sqrt{s} - |\xi'|^2}, \qquad \beta(s,\xi') = \sqrt{\sqrt{s} + |\xi'|^2}, \qquad \gamma(s,\xi') = \sqrt{|\xi'|^2 - \sqrt{s}},$$

we have

$$\begin{aligned}
e_r(t,x,y) &= \frac{1}{2\pi i} \lim_{\delta \to +0} \int_0^{t+\delta} \left\{ G_{r(s+i0)}(x,y) - G_{r(s-i0)}(x,y) \right\} ds \\
&= (2\pi)^{-n} \int_0^t ds \int_{|\xi'|^4 \le s} d\xi' \\
&\quad \times \left\{ \frac{e^{i(x'-y')\xi'}}{(\alpha - i\beta)^2} \left(\frac{e^{i\alpha(x_1+y_1)}}{2\alpha} - \frac{e^{-\beta x_1 + i\alpha y_1} + e^{i\alpha x_1 - \beta y_1}}{\alpha + i\beta} + \frac{e^{-\beta(x_1+y_1)}}{2i\beta} \right) \right. \\
&\qquad \left. - \frac{e^{i(x'-y')\xi'}}{(i\beta + \alpha)^2} \left(\frac{e^{-\beta(x_1+y_1)}}{2i\beta} - \frac{e^{-i\alpha x_1 - \beta y_1} + e^{-\beta x_1 - i\alpha y_1}}{i\beta - \alpha} + \frac{e^{-i\alpha(x_1+y_1)}}{-2\alpha} \right) \right\} \\
&\quad + (2\pi)^{-n} \int_0^t ds \int_{|\xi'|^4 \ge s} d\xi' \\
&\quad \times \left\{ \frac{e^{i(x'-y')\xi'}}{(i\gamma - i\beta)^2} \left(\frac{e^{-\gamma(x_1+y_1)}}{2i\gamma} - \frac{e^{-\beta x_1 - \gamma y_1} + e^{-\gamma x_1 - \beta y_1}}{i\gamma + i\beta} + \frac{e^{-\beta(x_1+y_1)}}{2i\beta} \right) \right. \\
&\qquad \left. - \frac{e^{i(x'-y')\xi'}}{(i\beta - i\gamma)^2} \left(\frac{e^{-\beta(x_1+y_1)}}{2i\beta} - \frac{e^{-\gamma x_1 - \beta y_1} + e^{-\beta x_1 - \gamma y_1}}{i\beta + i\gamma} + \frac{e^{-\gamma(x_1+y_1)}}{2i\gamma} \right) \right\}
\end{aligned}$$

$$= (2\pi)^{-n} \int_0^t ds \int_{|\xi'|^4 \le s} d\xi' e^{i(x'-y')\xi'}$$
$$\times \operatorname{Re}\left\{ \frac{1}{(\alpha - i\beta)^2} \left(\frac{e^{i\alpha(x_1+y_1)}}{\alpha} + \frac{e^{-\beta(x_1+y_1)}}{i\beta} - 2 \cdot \frac{e^{-\beta x_1 + i\alpha y_1} + e^{i\alpha x_1 - \beta y_1}}{\alpha + i\beta} \right) \right\}.$$

Let c_n be the area of the unit sphere in $\mathbb{R}^{n-1}$. Set $a = a(u) = \sqrt{1-u^2}$ and $b = b(u) = \sqrt{1+u^2}$. Changing the variables of integration and using $\alpha(s, s^{1/4}\xi') = s^{1/4}a(|\xi'|)$ and $\beta(s, s^{1/4}\xi') = s^{1/4}b(|\xi'|)$, we have

$$e_r(t,x,x) = (2\pi)^{-n} \int_0^t ds \int_{|\xi'|^4 \le s} d\xi' \operatorname{Re}\left\{ \frac{1}{(\alpha - i\beta)^2} \left(\frac{e^{i2\alpha x_1}}{\alpha} + \frac{e^{-2\beta x_1}}{i\beta} - \frac{4e^{(i\alpha-\beta)x_1}}{\alpha + i\beta} \right) \right\}$$
$$= (2\pi)^{-n} c_n \int_0^t ds \int_0^1 du$$
$$\times s^{n/4-1} u^{n-2} \operatorname{Re}\left\{ \frac{1}{(a-ib)^2} \left(\frac{e^{i2s^{1/4}ax_1}}{a} + \frac{e^{-2s^{1/4}bx_1}}{ib} - \frac{4e^{s^{1/4}(ia-b)x_1}}{a+ib} \right) \right\}$$
$$= (2\pi)^{-n} c_n t^{n/4} \int_0^1 s^{n-1} ds \int_0^1 du \tag{3.11}$$
$$\times u^{n-2} \operatorname{Re}\left\{ (a+ib)^2 \left(\frac{e^{i2t^{1/4}sax_1}}{a} + \frac{e^{-2t^{1/4}sbx_1}}{ib} - \frac{4e^{t^{1/4}s(ia-b)x_1}}{a+ib} \right) \right\}$$
$$= (2\pi)^{-n} c_n t^{n/4} (I_1 + I_2 + I_3).$$

Using $0 \le Xe^{-X} \le e^{-1}$ for $X \ge 0$, $|a+ib| = \sqrt{2}$ and $b \ge 1$ for $0 \le u \le 1$, we have

$$t^{1/4}x_1(|I_2| + |I_3|) \le e^{-1} \int_0^1 s^{n-2} ds \int_0^1 du\, 2u^{n-2} \left(\frac{1}{2b^2} + \frac{4}{|a+ib|b} \right) \le \frac{10e^{-1}}{(n-1)^2}.$$

Integration by parts and $a^2 - b^2 = -2u^2$ give

$$t^{1/4}x_1 I_1 = t^{1/4}x_1 \int_0^1 u^{n-2}\, du \int_0^1 ds\, s^{n-1} \operatorname{Re}\left\{ \frac{(a+ib)^2 e^{i2t^{1/4}sax_1}}{a} \right\}$$
$$= \int_0^1 u^{n-2}\, du \operatorname{Re}\left\{ \frac{(a+ib)^2 e^{i2t^{1/4}ax_1}}{i2a^2} - (n-1) \int_0^1 s^{n-2} \cdot \frac{(a+ib)^2 e^{i2t^{1/4}sax_1}}{i2a^2}\, ds \right\}$$
$$= \int_0^1 u^{n-2} \cdot \frac{b\cos(2t^{1/4}ax_1)}{a}\, du - \int_0^1 u^n \cdot \frac{\sin(2t^{1/4}ax_1)}{a^2}\, du$$
$$- (n-1) \int_0^1 s^{n-2}\, ds \int_0^1 u^{n-2} \cdot \frac{b\cos(2t^{1/4}sax_1)}{a}\, du$$
$$+ (n-1) \int_0^1 s^{n-2} ds \int_0^1 u^n \cdot \frac{\sin(2t^{1/4}sax_1)}{a^2}\, du$$
$$= J_1 + J_2 + J_3 + J_4.$$

It easily follows from $1 \le b \le \sqrt{2}$ for $0 \le u \le 1$ that

$$|J_1| \le \int_0^1 \frac{\sqrt{2}}{\sqrt{1-u^2}}\,du, \qquad |J_3| \le \int_0^1 \frac{\sqrt{2}}{\sqrt{1-u^2}}\,du.$$

J_2 and J_4 can be estimated by constants by the next lemma. Combining the above estimates, we get (3.10). □

Lemma 3.3.

$$\sup_{X>0}\left|\int_0^1 u^n \cdot \frac{\sin(X\sqrt{1-u^2})}{1-u^2}\,du\right| < \infty.$$

Proof. Changing the variable of integration, we have

$$\int_0^1 u^n \cdot \frac{\sin(X\sqrt{1-u^2})}{1-u^2}\,du = \int_0^X \left\{1-\left(\frac{v}{X}\right)^2\right\}^{(n-1)/2} \frac{\sin v}{v}\,dv = \int_0^X \frac{\sin v}{v}\,dv + V(X).$$

As is well known, $\int_0^X (\sin v)/v\,dv$ is bounded for $X > 0$. It is easily seen that $0 \le 1-(1-Y)^q \le (q+1)Y$ for $0 \le Y \le 1$ and $q > 0$, which gives

$$|V(X)| \le \int_0^X \frac{n+1}{2}\left(\frac{v}{X}\right)^2 \frac{1}{v}\,dv = \frac{n+1}{4}.$$

□

Proposition 3.4. *There exists a constant $C > 0$ such that*

$$\left|\int_0^R e_r(t,x,x)\,dx_1\right| \le Ct^{(n-1)/4}$$

for any $R > 0$, $t > 0$ and $x' \in \mathbb{R}^{n-1}$, where $x = (x_1, x')$.

Proof. Integrating (3.11) with respect to x_1, we have

$$\begin{aligned}
&\int_0^R e_r(t,x,x)\,dx_1 \\
&= (2\pi)^{-n} c_n t^{(n-1)/4} \int_0^1 s^{n-2} ds \int_0^1 du\, u^{n-2} \\
&\qquad \times \mathrm{Re}\left\{(a+ib)^2\left(\frac{e^{i2t^{1/4}saR}-1}{i2a^2} - \frac{e^{-2t^{1/4}sbR}-1}{i2b^2} - \frac{4(e^{t^{1/4}s(ia-b)R}-1)}{i(a+ib)^2}\right)\right\} \\
&= (2\pi)^{-n} c_n t^{(n-1)/4}(I_4 + I_5 + I_6).
\end{aligned}$$

Since $|a+ib|^2 = 2$ and $b \ge 1$ for $0 \le u \le 1$, we have

$$|I_5| + |I_6| \le \int_0^1 s^{n-2} ds \int_0^1 u^{n-2}\left(\frac{|a+ib|^2}{2b^2} + 4\cdot 2\right) du = \frac{9}{(n-1)^2}.$$

Use of $a^2 - b^2 = -2u^2$ gives

$$I_4 = \int_0^1 s^{n-2} ds \int_0^1 du \left\{ u^{n-2} \cdot \frac{b\left(\cos(2t^{1/4}saR) - 1\right)}{a} - u^n \cdot \frac{\sin(2t^{1/4}saR)}{a^2} \right\} = J_5 + J_6.$$

It easily follows from $1 \le b \le \sqrt{2}$ for $0 \le u \le 1$ that

$$|J_5| \le \frac{1}{n-1} \int_0^1 \frac{2\sqrt{2}}{\sqrt{1-u^2}} du.$$

Finally, J_6 is estimated by a constant by Lemma 3.3. Combining the above estimates, we get the proposition. □

Acknowledgments

This research was partly supported by Sato Fund, Nihon University School of Dentistry.

References

1. S. Agmon, Asymptotic formulas with remainder estimates for eigenvalues of elliptic operators, Arch. Rational Mech. Anal. **28** (1968), 165-183.
2. J. Brüning, Zur abshatzung der spektral-function elliptisher operatoren, Math. Z. **137** (1974), 75-85.
3. Y. Miyazaki, The eigenvalue distribution of elliptic operators with Hölder continuous coefficients II, Osaka J. Math. **30** (1993), 267-301.
4. Y. Miyazaki, Asymptotic behavior of spectral functions of elliptic operators with Hölder continuous coefficients, J. Math. Soc. Japan **49** (1997), 539-563.
5. Y. Miyazaki, Behavior of the spectral function for the Laplacian near the boundary, Transactions of Nihon University School of Dentistry (General), **29** (2001), 81-86.
6. Pham The Lai, Meilleures estimations asymptotiques des restes de la fonction spectrale et des valeurs propres relatifs au laplacien, Math. Scand. **48** (1981), 5-38.
7. R. T. Seeley, A sharp asymptotic remainder estimate for the eigenvalues of the Laplacian in a domain of $\mathbb{R}^3$, Adv. in Math. **29** (1978), 244-269.
8. R. T. Seeley, An estimate near the boundary for the spectral function of the Laplace operator, Amer. J. Math. **102** (1980), 869-902.
9. J. Tsujimoto, On the asymptotic behavior of spectral functions of elliptic operators, Japan. J. Math. **8** (1982), 177-210.