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Dirichlet problem for ordinary differential operators and the Laplacian

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Abstract

Let A be the divergence-form elliptic operator of order $2m$ defined in a domain Ω of $\mathbb{R}^n$ with smooth bounded boundary. Let $1 < p < \infty$. It is known that under suitable conditions the bounded inverse of $(A - \lambda)$ exists for λ in a certain sector if A is regarded as a bounded operator from the L^p Sobolev space of order m associated with the Dirichlet condition to the L^p Sobolev space of order $-m$, and that its proof is reduced to that of a proposition, in which it is assumed that A is a homogenuous operator with constant coefficients and that Ω is the half space. This paper is intended to give another proof of this proposition for $n = 1$ or $m = 1$.

Key words : elliptic operator, divergence form, resolvent, L^p theory, Dirichlet boundary value problem, Fourier multiplier, Poisson kernel

1. Introduction

Let Ω be a domain in the n-dimensional Euclidean space $\mathbb{R}^n$ and let A be the $2m$th elliptic operator of divergence form defined by

$$Au(x) = \sum_{|\alpha|\le m,\, |\beta|\le m} D^\alpha (a_{\alpha\beta}(x) D^\beta u(x)),$$

where $x = (x_1, \ldots, x_n)$ is a generic point and $\alpha = (\alpha_1, \ldots, \alpha_n)$ and β are multi-indices and

$$D = (D_1, \ldots, D_n), \qquad D_j = -\sqrt{-1}\frac{\partial}{\partial x_j} \quad (j = 1, \ldots, n).$$

We assume that the principal symbol $a(x, \xi)$ of A satisfies

$$a(x,\xi) = \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)\xi^{\alpha+\beta} \ge \delta_A |\xi|^{2m}$$

for any $x \in \Omega$ and $\xi \in \mathbb{R}^n$ with some $\delta_A > 0$ and that the coefficients $a_{\alpha\beta}$ are measurable and bounded. We set

$$\zeta_A = (n, m, \delta_A, M_A), \qquad M_A = \max_{|\alpha|,|\beta|\le m} \|a_{\alpha\beta}\|_{L^\infty(\Omega)},$$

$$\omega_A(\varepsilon) = \max_{|\alpha|=|\beta|=m} \sup_{x\in\Omega} \sup_{|h|\le\varepsilon,\, x+h\in\Omega} |a_{\alpha\beta}(x+h) - a_{\alpha\beta}(x)|,$$

$$\Lambda(R,\theta) = \{z \in \mathbb{C} : |z| \ge R,\ \theta \le \arg z \le 2\pi - \theta\}.$$

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For a fixed $p \in (1, \infty)$ we regard A as a bounded operator:

$$A: H_0^{m,p}(\Omega) \to H^{-m,p}(\Omega).$$

Here $H^{\sigma,p}(\Omega)$ with $\sigma \in \mathbb{R}$ is the L^p Sobolev space of order σ and $H_0^{\sigma,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$, the space of C^∞ functions with compact support, in $H^{\sigma,p}(\Omega)$.

When $p = 2$, by associating A with a sesquilinear form we can apply the theory of Hilbert spaces. In particular, Gårding's inequality and Lax-Milgram's theorem show that $(A - \lambda)$ has a bounded inverse for λ in an appropriate region of the complex plane $\mathbb{C}$. It is important to derive a similar result for a general $p \in (1, \infty)$. In [1, 2, 3] the following theorem was established.

Theorem 1.1. *Let $p \in (1, \infty)$ and $\theta \in (0, \pi/2)$. Suppose that the coefficients $a_{\alpha\beta}$ with $|\alpha| = |\beta| = m$ are uniformly continuous in the closure of Ω and that one of the following is satisfied:*

(i) $\Omega = \mathbb{R}^n$,

(ii) $\Omega = \mathbb{R}_+^n$, *where* $\mathbb{R}_+^n = \{(x', x_n) : x' \in \mathbb{R}^{n-1},\ x_n > 0\}$,

(iii) Ω *is a domain with C^{m+1} bounded boundary.*

Then we conclude that there are $R = R(p, \theta, \zeta_A, \omega_A) \geq 1$ and $K = K(p, \theta, \zeta_A) > 0$ such that the resolvent $(A - \lambda)^{-1}$ exists and satisfies

$$||(A - \lambda)^{-1}||_{H^{-i,p}(\Omega) \to H^{j,p}(\Omega)} \leq K|\lambda|^{-1+(i+j)/2m} \tag{1.1}$$

for $0 \leq i \leq m$, $0 \leq j \leq m$ and $\lambda \in \Lambda(R, \theta)$.

Theorem 1.1 was proved when $\Omega = \mathbb{R}^n$ in [1], when $m = 1$ in [2], and for the general case in [3]. According to [2], the proof of Theorem 1.1 is reduced to that of the following proposition.

Proposition 1.2. *In addition to the assumptions in Theorem 1.1 suppose that A is a homogeneous operator with constant coefficients and $\Omega = \mathbb{R}_+^n$. Then the conclusion of Theorem 1.1 is valid with $R = 1$.*

Proposition 1.2 was proved when $m = 1$ by the reflection method in [2] and for the general case in [3]. The purpose of this paper is to give another proof of Proposition 1.2 when $n = 1$ (ordinary differentail operators) or $m = 1$ (second-order operators).

Let us suppose the assumptions in Proposition 1.2. Let $\lambda \in \mathbb{C} \setminus [0, \infty)$ and $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$ and $x_n > 0$. Showing the existence of the resolvent is equivalent to finding a solution $u \in H_0^{m,p}(\mathbb{R}_+^n)$ to the equation

$$\begin{cases} (A - \lambda)u = f \\ D_n^{l-1} u(x', 0) = 0 \quad (l = 1, \ldots, m) \end{cases} \tag{1.2}$$

for a given $f \in H^{-m,p}(\mathbb{R}^n_+)$. Estimate (1.1) is equivalent to

$$\|u\|_{H^{j,p}(\mathbb{R}^n_+)} \leq K'|\lambda|^{-1+(i+j)/2m}\|f\|_{H^{-i,p}(\mathbb{R}^n_+)} \tag{1.3}$$

for $0 \leq i \leq m$ and $0 \leq j \leq m$ with some constant K'. Equation (1.2) can be solved by setting $u = v_\lambda + w_\lambda$ with v_λ, $w_\lambda \in H^{m,p}(\mathbb{R}^n_+)$ satisfying the equations

$$(A - \lambda)v_\lambda = f \tag{1.4}$$

and

$$\begin{cases} (A-\lambda)w_\lambda = 0 \\ D_n^{l-1}w_\lambda(x', 0) = g_l(x') \quad (l = 1, \dots, m) \end{cases} \tag{1.5}$$

with $g_l(x') = -D_n^{l-1}v(x', 0)$. We note that $g_l \in B_{pp}^{m-l+1-1/p}(\mathbb{R}^{n-1})$ if $n \geq 2$ and $g_l \in \mathbb{C}$ if $n = 1$. The mapping $\{g_l\}_{1 \leq l \leq m} \mapsto w_\lambda$ is called the Poisson operator.

As shown in [3], it is easy to solve equation (1.4) and get estimate (1.3) with u replaced by v_λ. So it remains to solve (1.5) with norm estimates. When $n = 1$ or $m = 1$, the problem is simpler than that of the general case because the algebraic equation

$$a(x, \xi', \xi_n) - \lambda = 0 \tag{1.6}$$

for ξ_n has no multiple root. The case $n = 1$ is treated by evaluating deteminants in Section 2.

In Section 3 we consider the case of $m = 1$ and $n \geq 2$. As stated in [2], a linear transformation reduces the problem to the case of the Laplacian. Then w_λ satisfying (1.5) is expressed by a Fourier multiplier in the x'-space with parameter x_n. The norms of w_λ in the L^p Sobolev spaces are estimated by Mihlin's multiplier theorem [4, Theorem 6.16] and a theorem of L^p boundedness for the integral operator associated with the Fourier multiplier (Lemma 3.3). In [3] we succeeded in proving Proposition 1.2 for the general case by developing the idea of Section 3 and using the formula for the Poisson operator by the Cauchy integral (see [5]), which enables us to deal with the case where (1.6) has multiple roots.

When $p = 2$, the norms of w_λ for the Laplacian can be estimated more easily than in Section 3 with help of Parseval's formula. This method is presented in Section 4.

2. Ordinary differential operators

In this section we consider the case $n = 1$ and give another proof of Proposition 1.2. So we have $\Omega = \mathbb{R}_+ = (0, \infty)$ and

$$A = a\left(\frac{1}{\sqrt{-1}}\frac{d}{dx}\right)^{2m} = (-1)^m a \frac{d^{2m}}{dx^{2m}},$$

where a is a constant. Clearly $\delta_A = M_A = a$. Equation (1.5) is rewritten as

$$\begin{cases} (A-\lambda)w_\lambda = 0 \\ w_\lambda^{(l-1)}(0) = g_l \quad (l = 1, \dots, m) \end{cases} \tag{2.1}$$

with $g_l \in \mathbb{C}$, where we replaced $(\sqrt{-1})^{l-1}g_l$ by g_l. The argument in [3] shows that the proof of Proposition 1.2 is reduced to that of the following lemma.

Lemma 2.1. *Let $p \in (1,\infty)$ and $\theta \in (0,\pi/2)$. Then there exists $C = C(p,\theta,\zeta_A)$ such that the solution w_λ to equation* (2.1) *satisfies*

$$\|w_\lambda\|_{H^{j,p}(\mathbb{R}_+)} \le C|\lambda|^{j/2m-1/2mp}\sum_{l=1}^{m}|\lambda|^{(1-l)/2m}|g_l|$$

for $1 \le j \le m$ and $\lambda \in \Lambda(1,\theta)$.

Proof. Without loss of generality we may assume $a = 1$. Set

$$\tau_j(\lambda) = \sqrt{-1}\,e^{(2j-1)\pi\sqrt{-1}/2m}(-\lambda)^{1/2m} \quad (j = 1,\ldots,2m),$$

where $z^{1/2m}$ stands for the branch of the power which is positive when $z > 0$. It is easy to see that $-\sqrt{-1}\,\tau_j(\lambda)$ $(j = 1,\ldots,2m)$ are the roots of $z^{2m} - \lambda = 0$ and that

$$\operatorname{Re}\tau_j(\lambda) < 0 \;\; (j = 1,\ldots,m) \quad \text{and} \quad \operatorname{Re}\tau_j(\lambda) > 0 \;\; (j = m+1,\ldots,2m).$$

So the solution w_λ to (2.1) can be found in the form

$$w_\lambda(x) = \sum_{j=1}^{m} c_j e^{\tau_j(\lambda)x},$$

where c_j's satisfy

$$\sum_{j=1}^{m}\tau_j(\lambda)^{l-1}c_j = g_l \qquad (l = 1,\ldots,m),$$

which is equivalent to

$$Tc = g,$$

where $T = \left(\tau_j(\lambda)^{k-1}\right)_{k,j}$, $c = {}^t(c_1,\ldots,c_m)$ and $g = {}^t(g_1,\ldots,g_m)$. Let S_l $(l = 1,\ldots,m)$ be the matrix obtained from T by replacing the lth column with g, that is, $S_l = \left(s_{l,kj}\right)_{k,j}$ with $s_{l,kj} = \tau_j(\lambda)^{k-1}$ for $j \ne l$ and $s_{l,kj} = g_k$ for $j = l$. Then Cramer's formula gives

$$c_l = \frac{\det S_l}{\det T}.$$

Since $|\tau_k(\lambda) - \tau_j(\lambda)| \ge |e^{\sqrt{-1}\pi/m} - 1|\,|\lambda|^{1/2m}$ for $k \ne j$ and $\det T$ is the determinant of Vandermonde's matrix, we have

$$|\det T| = \prod_{k<j}|\tau_k(\lambda) - \tau_j(\lambda)| \ge c(|\lambda|^{1/2m})^{m(m-1)/2} = c|\lambda|^{(m-1)/4}.$$

On the other hand,

$$\det S_l = \sum_\sigma \operatorname{sgn}(\sigma) \prod_{j=1}^m s_{l,\sigma(j)j} = \sum_\sigma \operatorname{sgn}(\sigma) \left(\prod_{j=1}^m \tau_j(\lambda)^{\sigma(j)-1} \right) \frac{g_{\sigma(l)}}{\tau_l(\lambda)^{\sigma(l)-1}},$$

where the sum is taken over all the permutations of order n. Using the equality

$$\sum_{j=1}^m (\sigma(j)-1)/2m = \sum_{j=1}^m (j-1)/2m = (m-1)/4$$

which follows from $\{\sigma(1),\ldots,\sigma(m)\} = \{1,\ldots,m\}$, and noting $\cup_\sigma\{\sigma(l)\} = \{1,\ldots,m\}$, we have

$$\begin{aligned} |\det S_l| &\le \sum_\sigma \left(\prod_{j=1}^m |\lambda|^{(\sigma(j)-1)/2m} \right) |\lambda|^{(1-\sigma(l))/2m} |g_{\sigma(l)}| \\ &\le (m-1)! \sum_{k=1}^m |\lambda|^{(m-1)/4+(1-k)/2m} |g_k|. \end{aligned}$$

Therefore we get

$$|c_l| \le C \sum_{k=1}^m |\lambda|^{(1-k)/2m} |g_k|.$$

Since $\operatorname{Re}\tau_j(\lambda) \le -(\sin\frac{\theta}{2m})|\lambda|^{1/2m}$, we have

$$\begin{aligned} \|w_\lambda^{(j)}\|_{L^p(\mathbb{R}_+)} &\le \sum_{l=1}^m \|c_l \tau_l(\lambda)^j e^{\tau_l(\lambda)x}\|_{L^p(\mathbb{R}_+)} \\ &\le \sum_{l=1}^m |c_l| \cdot |\lambda|^{j/2m} (p|\operatorname{Re}\tau_j(\lambda)|)^{-1/p} \\ &\le C \sum_{l=1}^m \sum_{k=1}^m |\lambda|^{(1-k)/2m} |g_k|\, |\lambda|^{j/2m-1/2mp}, \end{aligned}$$

from which the lemma follows. □

3. Laplacian

In this section we consider the case of $m = 1$ and $n \ge 2$ and give another proof of Proposition 1.2. For sake of simplicity we use the following notation:

$$x = (x', t), \quad x' = (x_1, \ldots, x_{n-1}), \quad D_n = -\sqrt{-1}\,\frac{\partial}{\partial t}, \quad \xi' = (\xi_1, \ldots, \xi_{n-1}).$$

As stated in Introduction, we may assume $A = -\Delta$. So equation (1.5) is rewritten as

$$\begin{cases} (-\Delta - \lambda) w_\lambda = 0 \\ w_\lambda(x', 0) = g(x') \end{cases} \tag{3.1}$$

with $g \in B_{pp}^{1-1/p}(\mathbb{R}^{n-1})$. Moreover, by virtue of the argument in [3] the proof of Proposition 1.2 is reduced to those of Lemmas 3.2, 3.4 and 3.5 below.

By the partial Fourier transformation $\mathcal{F}'$, which is defined by

$$\mathcal{F}'u(\xi', t) = \int_{\mathbb{R}^{n-1}} e^{-\sqrt{-1}x'\xi'} u(x', t)\, dx',$$

equation (3.1) is transformed into

$$\begin{cases} \left(-\frac{\partial^2}{\partial t^2} + |\xi'|^2 - \lambda\right) \mathcal{F}'w_\lambda(\xi', t) = 0 \\ \mathcal{F}'w_\lambda(\xi', 0) = \mathcal{F}g(\xi'), \end{cases}$$

where $\mathcal{F}g$ denotes the Fourier transform of g. Since $\mathcal{F}'w_\lambda(\xi', t) \to 0$ as $t \to \infty$, we have

$$\mathcal{F}'w_\lambda(\xi', t) = e^{-t\tau(\xi', \lambda)} \mathcal{F}g(\xi')$$

with $\tau(\xi', \lambda) = \sqrt{|\xi'|^2 - \lambda}$, where $\sqrt{z}$ stands for the branch of the square root which is positive for $z > 0$, and therefore

$$\begin{aligned} w_\lambda(x', t) &= (2\pi)^{1-n} \int_{\mathbb{R}^{n-1}} e^{\sqrt{-1}\,x'\xi'} e^{-t\tau(\xi', \lambda)} \mathcal{F}g(\xi')\, d\xi' \\ &= (2\pi)^{1-n} \iint_{\mathbb{R}^{2n-2}} e^{\sqrt{-1}(x'-y')\xi'} e^{-t\tau(\xi', \lambda)} g(y')\, dy'\, d\xi'. \end{aligned} \tag{3.2}$$

Lemma 3.1. *The inequality*

$$\mathrm{Re}\sqrt{s - \lambda} \geq \frac{d(\lambda)}{8|\lambda|}(s^{1/2} + |\lambda|^{1/2}) \tag{3.3}$$

holds for $s \geq 0$ and $\lambda \in \mathbb{C} \setminus [0, \infty)$, where $d(\lambda) = \mathrm{dis}(\lambda, [0, \infty))$. Moreover, for any $\theta \in (0, \pi/2)$ and $\alpha \in \mathbb{N}^{n-1}$ there exist $c = c(\theta)$ and $C = C(\theta, \alpha)$ such that

$$\mathrm{Re}\,\tau(\xi', \lambda) \geq c(|\xi'| + |\lambda|^{1/2}), \tag{3.4}$$

$$|\partial_{\xi'}^\alpha \tau(\xi', \lambda)| \leq C(|\xi'| + |\lambda|^{1/2})^{1-|\alpha|}, \tag{3.5}$$

$$|\partial_{\xi'}^\alpha e^{-t\tau(\xi', \lambda)}| \leq C(|\xi'| + |\lambda|^{1/2})^{-|\alpha|} e^{-ct(|\xi'|+|\lambda|^{1/2})} \tag{3.6}$$

for $t > 0$, $\xi' \in \mathbb{R}^{n-1}$ and $\lambda \in \Lambda(1, \theta)$.

Proof. Note that $\sqrt{x + \sqrt{-1}y} = X + \sqrt{-1}Y$ with $x, y, X, Y \in \mathbb{R}$ implies $2X^2 = x + \sqrt{x^2 + y^2}$. Hence setting $X = \mathrm{Re}\sqrt{s - \lambda}$, we have

$$2X^2 = (s - \lambda_R) + \sqrt{(s - \lambda_R)^2 + \lambda_I^2},$$

where $\lambda = \lambda_R + \sqrt{-1}\lambda_I$ with $\lambda_R, \lambda_I \in \mathbb{R}$. There are four cases to evaluate X. When $\lambda_R \leq 0$,

$$2X^2 \geq \sqrt{(s + |\lambda_R|)^2 + \lambda_I^2} \geq \sqrt{s^2 + |\lambda|^2}.$$

When $0 \le 2\lambda_R \le s$,

$$2X^2 \ge \sqrt{(2^{-1}s)^2 + \lambda_I^2} \ge \sqrt{8^{-1}(s^2 + |\lambda|^2)}.$$

When $0 \le \lambda_R \le s \le 2\lambda_R$,

$$2X^2 \ge d(\lambda) \ge \left(\frac{d(\lambda)}{|\lambda|}\right)^2 |\lambda| \ge \left(\frac{d(\lambda)}{|\lambda|}\right)^2 \sqrt{8^{-1}(s^2 + |\lambda|^2)}.$$

When $0 \le s \le \lambda_R$,

$$2X^2 \ge \frac{\lambda_I^2}{(\lambda_R - s) + \sqrt{(\lambda_R - s)^2 + \lambda_I^2}} \ge \frac{d(\lambda)^2}{2|\lambda|} \ge \frac{d(\lambda)^2}{2|\lambda|^2}\sqrt{2^{-1}(s^2 + |\lambda|^2)}.$$

Summing up, we get $2X^2 \ge 4^{-1}(d(\lambda)/|\lambda|)^2\sqrt{s^2 + |\lambda|^2}$, from which (3.3) follows.

Inequality (3.4) with $c = 8^{-1}\sin\theta$ is an immediate consequence of (3.3).

We have $\partial_{\xi_j}\tau = \xi_j\tau^{-1}$ and $\partial_{\xi_l}\partial_{\xi_j}\tau = \delta_{jl}\tau^{-1} + \xi_j \cdot (-1)\tau^{-2} \cdot \xi_l\tau^{-1}$. So induction on α leads to

$$\partial_{\xi'}^{\alpha}\tau = \tau^{1-|\alpha|}\sum_{\beta \le \alpha} C_{\alpha\beta}\xi'^{\beta}\tau^{-|\beta|}.$$

This combined with $|\tau(\xi', \lambda)| \ge c(|\xi'| + |\lambda|^{1/2})$, which follows from (3.4), gives (3.5).

Simple calculation shows

$$\partial_{\xi'}^{\alpha}e^{-t\tau(\xi',\lambda)} = \sum_{\alpha_1 + \cdots + \alpha_k = \alpha} C_{\alpha_1\cdots\alpha_k}t^k\partial_{\xi'}^{\alpha_1}\tau(\xi', \lambda)\cdots\partial_{\xi'}^{\alpha_k}\tau(\xi', \lambda) \cdot e^{-t\tau(\xi',\lambda)}.$$

So (3.4) and (3.5) give

$$\left|\partial_{\xi'}^{\alpha}e^{-t\tau(\xi',\lambda)}\right| \le \sum_{k \le |\alpha|} C_k t^k(|\xi'| + |\lambda|^{1/2})^{k-|\alpha|}e^{-ct(|\xi'|+|\lambda|^{1/2})}.$$

Then use of the inequality $s^k e^{-as} \le k^k(ae)^{-k}$ for $s > 0$, $a > 0$ and $k > 0$ yields (3.6). □

Lemma 3.2. *Let $p \in (1, \infty)$ and $\theta \in (0, \pi/2)$. Then there exists $C = C(p, \theta, \zeta_A)$ such that the solution w_λ to* (3.1) *satisfies*

$$||w_\lambda||_{L^p(\mathbb{R}^n_+)} \le C|\lambda|^{-1/2p}||g||_{L^p(\mathbb{R}^{n-1})} \quad \textit{for } \lambda \in \Lambda(1, \theta).$$

Proof. By (3.6) we have

$$|\xi'|^{|\alpha|}\left|\partial_{\xi'}^{\alpha}e^{-t\tau(\xi',\lambda)}\right| \le Ce^{-ct|\lambda|^{1/2}}$$

for $|\alpha| \le [n/2] + 1$. So Mihlin's multiplier theorem yields

$$||w_\lambda(\,\cdot\,, t)||_{L^p(\mathbb{R}^{n-1})} \le Ce^{-ct|\lambda|^{1/2}}||g||_{L^p(\mathbb{R}^{n-1})}.$$

Then the lemma follows from $||e^{-ct|\lambda|^{1/2m}}||_{L^p(\mathbb{R}_+)} = C|\lambda|^{-1/2p}$. □

Lemma 3.3. *Let $p \in (1,\infty)$. Assume that $K(x',t)$ satisfies*

$$|K(x',t)| \leq C_1 t^{-n}(1+|x'|/t)^{\sigma-n} \tag{3.7}$$

with some constants $C_1 > 0$ and $\sigma \in (0,1/p)$, and set

$$u(x',t) = \int_{\mathbb{R}^{n-1}} K(y',t)\{g(x'-y') - g(x')\}\, dy'$$

for $g \in B_{pp}^{1-1/p}(\mathbb{R}^{n-1})$. Then $u \in L^p(\mathbb{R}^n_+)$ and there exists $C = C(n,p,\sigma)$ such that

$$\|u\|_{L^p(\mathbb{R}^n_+)} \leq CC_1\|g\|_{B_{pp}^{1-1/p}(\mathbb{R}^{n-1})}.$$

Proof. By (3.7) we have

$$\begin{aligned} t^{1/p}\|u(\,\cdot\,,t)\|_{L^p(\mathbb{R}^{n-1})} &\leq \int_{\mathbb{R}^{n-1}} t^{1/p}|K(-y',t)|\cdot\|\Delta_{y'}g\|_{L^p(\mathbb{R}^{n-1})}\, dy' \\ &\leq C_1\int_{\mathbb{R}^{n-1}} \left(\frac{|y'|}{t}\right)^{n-1/p}\left(1+\frac{|y'|}{t}\right)^{\sigma-n}\frac{\|\Delta_{y'}g\|_{L^p(\mathbb{R}^{n-1})}}{|y'|^{1-1/p}}\frac{dy'}{|y'|^{n-1}}. \end{aligned}$$

We can regard the last integral as a bounded operator from $L^p(\mathbb{R}^{n-1},|y'|^{1-n}dy')$ to $L^p(\mathbb{R}_+,t^{-1}dt)$ with kernel $H(y',t) = (|y'|/t)^{n-1/p}(1+|y'|/t)^{\sigma-n}$ for

$$\begin{aligned} \int_{\mathbb{R}^{n-1}} H(y',t)|y'|^{1-n}dy' &= \int_{\mathbb{R}^{n-1}} |y'|^{1-1/p}(1+|y'|)^{\sigma-n}dy' \\ &= C\int_0^\infty s^{n-1-1/p}(1+s)^{\sigma-n}ds < \infty, \\ \int_0^\infty H(y',t)t^{-1}dt &= \int_0^\infty s^{n-1-1/p}(1+s)^{\sigma-n}ds < \infty. \end{aligned}$$

Since $\|u\|_{L^p(\mathbb{R}^n_+)} = \|t^{1/p}\|u(\,\cdot\,,t)\|_{L^p(\mathbb{R}^{n-1})}\|_{L^p(\mathbb{R}_+,t^{-1}dt)}$, we get the lemma. □

Lemma 3.4. *Let $p \in (1,\infty)$, $\theta \in (0,\pi/2)$ and $1 \leq k < n$. Then there exists $C = C(p,\theta,\zeta_A)$ such that the solution w_λ to (3.1) satisfies*

$$\|D_k w_\lambda\|_{L^p(\mathbb{R}^n_+)} \leq C\|g\|_{B_{pp}^{1-1/p}(\mathbb{R}^{n-1})} \quad \text{for } \lambda \in \Lambda(1,\theta).$$

Proof. Set

$$K_\lambda(x',t) = (2\pi)^{1-n}\int_{\mathbb{R}^{n-1}} e^{\sqrt{-1}x'\xi'}\xi_k e^{-t\tau(\xi',\lambda)}\, d\xi'.$$

Then we have

$$\begin{aligned} D_k w_\lambda(x',t) &= \int_{\mathbb{R}^{n-1}} K_\lambda(x'-y',t)g(y')\, dy' \\ &= \int_{\mathbb{R}^{n-1}} K_\lambda(y',t)\{g(x'-y') - g(x')\}\, dy', \end{aligned}$$

where we used $\int_{\mathbb{R}^{n-1}} K_\lambda(x',t)\,dx' = \xi_k e^{-t\tau(\xi',\lambda)}\big|_{\xi'=0} = 0$. Now we shall show that for any $\sigma \in (0,1)$ there exists $C_1 = C_1(p,\sigma)$ such that $K_\lambda(x',t)$ satisfies inequality (3.7) with $K = K_\lambda$. By (3.6) we have

$$|K_\lambda(x',t)| \le C \int_{\mathbb{R}^{n-1}} |\xi'| e^{-ct|\xi'|} d\xi' \le Ct^{-n} \int_{\mathbb{R}^{n-1}} |\xi'| e^{-c|\xi'|}\, d\xi' \le Ct^{-n}. \tag{3.8}$$

On the other hand, integration by parts gives

$$\begin{aligned} x_j^n K_\lambda(x',t) &= C \int_{\mathbb{R}^{n-1}} e^{\sqrt{-1}x'\xi'} \frac{\partial^n}{\partial \xi_j^n} \left(\xi_k e^{-t\tau(\xi',\lambda)}\right) d\xi' \\ &= C \int_{\mathbb{R}^{n-1}} (e^{\sqrt{-1}x'\xi'} - 1) \frac{\partial^n}{\partial \xi_j^n} \left(\xi_k e^{-t\tau(\xi',\lambda)}\right) d\xi' \end{aligned}$$

for $1 \le j < n$. Using the inequality $|e^{\sqrt{-1}s} - 1| \le 2|s|^\sigma$ for $s \in \mathbb{R}$ and $\sigma \in (0,1)$, we have

$$\begin{aligned} |x_j|^n |K_\lambda(x',t)| &\le C \int_{\mathbb{R}^{n-1}} |x'|^\sigma |\xi'|^\sigma (|\xi'| + |\lambda|^{1/2})^{1-n} e^{-ct(|\xi'|+|\lambda|^{1/2})}\, d\xi' \\ &\le C|x'|^\sigma \int_{\mathbb{R}^{n-1}} |\xi'|^{\sigma+1-n} e^{-ct|\xi'|}\, d\xi' \le C|x'|^\sigma t^{-\sigma}, \end{aligned}$$

where it should be noted that the last integral is integrable for $\sigma + 1 - n > 1 - n$. Therefore

$$|K_\lambda(x',t)| \le C|x'|^{\sigma-n} t^{-\sigma}. \tag{3.9}$$

Since $|x'| \le t$ implies $t^{-n}(1+|x'|/t)^{\sigma-n} \ge 2^{\sigma-n}t^{-n}$, and $|x'| \ge t$ implies $t^{-n}(1+|x'|/t)^{\sigma-n} = (1+|x'|/t)^\sigma (t+|x'|)^{-n} \ge (|x'|/t)^\sigma (2|x'|)^{-n}$, inequalities (3.8) and (3.9) show that $K_\lambda(x',t)$ satisfies (3.7) with $K = K_\lambda$.

Finally, the lemma follows from Lemma 3.3. □

Lemma 3.5. *Let $p \in (1,\infty)$ and $\theta \in (0,\pi/2)$. Then there exists $C = C(p,\theta,\zeta_A)$ such that the solution w_λ to (3.1) satisfies*

$$\|D_n w_\lambda\|_{L^p(\mathbb{R}^n_+)} \le C\left(\|g\|_{B^{1-1/p}_{pp}(\mathbb{R}^{n-1})} + |\lambda|^{1/2-1/2p} \|g\|_{L^p(\mathbb{R}^{n-1})}\right) \quad \textit{for } \lambda \in \Lambda(1,\theta).$$

Proof. Set

$$K_\lambda(x',t) = \sqrt{-1}(2\pi)^{1-n} \int_{\mathbb{R}^{n-1}} e^{\sqrt{-1}x'\xi'} \tau(\xi',\lambda) e^{-t\tau(\xi',\lambda)}\, d\xi'.$$

Then we have

$$\begin{aligned} D_n w_\lambda(x',t) &= \int_{\mathbb{R}^{n-1}} K_\lambda(x'-y',t) g(y')\, dy' \\ &= \int_{\mathbb{R}^{n-1}} K_\lambda(y',t)\{g(x'-y') - g(x')\}\, dy' + J_\lambda(t) g(x'), \end{aligned} \tag{3.10}$$

where

$$J_\lambda(t) = \int_{\mathbb{R}^{n-1}} K_\lambda(x',t)\, dx' = \sqrt{-1}\tau(0,\lambda) e^{-t\tau(0,\lambda)}.$$

The last integral in (3.10) is estimated in the same way as in the proof of Lemma 3.4. The last term in (3.10) is evaluated by $\|J_\lambda\|_{L^p(\mathbb{R}_+)} \le C|\lambda|^{1/2-1/2p}$. Therefore the lemma follows. □

4. Simple proof for the Laplacian when p = 2

When $p = 2$, we do not need Mihlin's multiplier theorem and Lemma 3.3 to prove Lemmas 3.2, 3.4 and 3.5. In fact, we can prove these lemmas as follows.

Using Parseval's formula, Fubini's theorem and Lemma 3.1, we get from (3.2)

$$\begin{aligned}\|w_\lambda\|^2_{L^2(\mathbb{R}^n_+)} &= \int_0^\infty dt \int_{\mathbb{R}^{n-1}} |w_\lambda(x', t)|^2\, dx' = \int_0^\infty dt \int_{\mathbb{R}^{n-1}} \left|e^{-t\tau(\xi',\lambda)} \mathcal{F}g(\xi')\right|^2 d\xi' \qquad (4.1)\\ &= \int_{\mathbb{R}^{n-1}} \frac{1}{2\mathrm{Re}\,\tau(\xi',\lambda)} |\mathcal{F}g(\xi')|^2\, d\xi'\\ &\le C|\lambda|^{-1/2} \int_{\mathbb{R}^{n-1}} |\mathcal{F}g(\xi')|^2\, d\xi' \le C|\lambda|^{-1/2} \|g\|^2_{L^2(\mathbb{R}^{n-1})}.\end{aligned}$$

Differentiation of (3.2) yields the formulas for $D_j w_\lambda$ $(1 \le j < n)$ and $D_n w_\lambda$ by the Fourier multipliers with symbols $\xi_j e^{-t\tau(\xi',\lambda)}$ $(1 \le j < n)$ and $\sqrt{-1}\tau(\xi',\lambda)e^{-t\tau(\xi',\lambda)}$ respectively. In the same way as in (4.1) we get

$$\begin{aligned}\|D_j w_\lambda\|^2_{L^2(\mathbb{R}^n_+)} &= \int_{\mathbb{R}^{n-1}} \frac{|\xi_j|^2}{2\mathrm{Re}\,\tau(\xi',\lambda)} |\mathcal{F}g(\xi')|^2\, d\xi'\\ &\le C \int_{\mathbb{R}^{n-1}} |\xi_j||\mathcal{F}g(\xi')|^2\, d\xi' \le C\|g\|^2_{H^{1/2}(\mathbb{R}^{n-1})}\end{aligned}$$

for $1 \le j < n$ and

$$\begin{aligned}\|D_n w_\lambda\|^2_{L^2(\mathbb{R}^n_+)} &= \int_{\mathbb{R}^{n-1}} \frac{|\tau(\xi',\lambda)|^2}{2\mathrm{Re}\,\tau(\xi',\lambda)} |\mathcal{F}g(\xi')|^2\, d\xi'\\ &\le C \int_{\mathbb{R}^{n-1}} (|\xi'| + |\lambda|^{1/2}) |\mathcal{F}g(\xi')|^2\, d\xi'\\ &\le C \left(\|g\|_{H^{1/2}(\mathbb{R}^{n-1})} + |\lambda|^{1/2} \|g\|^2_{L^2(\mathbb{R}^{n-1})}\right).\end{aligned}$$

Acknowledgments

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