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Kernel estimates for pseudo-differential operators of negative order

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It is known that a pseudo-differential operator of negative order has an integral kernel that is evaluated by an L^1 function. However, most textbooks on pseudo-differential operators including [1] do not cover this theorem. Although Taylor [2, Proposition 7.2.2] gave the proof of this theorem, it is not easy to fill in the gap in his proof.

The aim of this note is to give a clear proof of this theorem by adopting the method of deriving the corresponding theorem for Fourier multipliers used in [3, Lemma 8.5.2]. As described in [3], this method usually requires showing that the integral kernel has no singularity related to the delta function. In our proof this fact is shown by the dominated convergence theorem. Compared to the proof of Kumano-go and Nagase [4, Proposition 2.1], our proof as well as the proof provided by Taylor [2] gives a precise exponent for the L^1 function.

Let $S^{-\sigma}(\mathbb{R}^n \times \mathbb{R}^n)$ with $\sigma \in \mathbb{R}$ denote the symbol class of order $-\sigma$ and type $(1,0)$, that is, we mean by $a(x,\xi) \in S^{-\sigma}(\mathbb{R}^n \times \mathbb{R}^n)$ that $|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \le C\langle\xi\rangle^{-\sigma-|\alpha|}$ with $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$ and some constant $C > 0$ independent of x and ξ. The pseudo-differential operator with symbol $a(x,\xi) \in S^{-\sigma}(\mathbb{R}^n \times \mathbb{R}^n)$ is defined by

$$a(x,D)f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\xi} \mathcal{F}f(\xi)\, d\xi, \qquad f \in \mathcal{S}(\mathbb{R}^n),$$

where $\mathcal{F}f(\xi) = \int_{\mathbb{R}^n} e^{-ix\xi} f(x)\, dx$. For $k \in \mathbb{N}_0 = \{0,1,2,\dots\}$ we set

$$|a|_k^{(-\sigma)} = \max_{|\alpha| \le k} \sup_{\substack{x\in\mathbb{R}^n \\ \xi\in\mathbb{R}^n}} |\partial_\xi^\alpha a(x,\xi)| \langle\xi\rangle^{\sigma+|\alpha|}.$$

Theorem 1. *Let $a(x,\xi) \in S^{-\sigma}(\mathbb{R}^n \times \mathbb{R}^n)$ with $\sigma > 0$. Then there exists a function $K(x,z)$ and a constant $C(n,\sigma)$ such that*

$$a(x,D)f(x) = \int_{\mathbb{R}^n} K(x, x-y)\, f(y)\, dy, \quad f \in \mathcal{S}(\mathbb{R}^n) \tag{1}$$

and

$$|K(x,z)| \le C(n,\sigma) |a|_{n+1}^{(-\sigma)} H_\sigma(z) \qquad \text{for } x \in \mathbb{R}^n,\ z \in \mathbb{R}^n, \tag{2}$$

where the function $H_\sigma(x)$ is defined by $H_\sigma(x) = |x|^{-n-1}$ for $|x| \ge 1$ and

$$H_\sigma(x) = \begin{cases} |x|^{\sigma-n} & (0 < \sigma < n), \\ 1 + \log_+ |x|^{-1} & (\sigma = n), \\ 1 & (\sigma > n) \end{cases}$$

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for $|x| \le 1$.

For the proof of Theorem 1 we need a lemma.

Lemma 2. *For* $\chi \in \mathcal{S}(\mathbb{R}^n)$ *with* $\chi(0) = 1$ *and* $0 < \varepsilon < 1$ *set* $\chi_\varepsilon(\xi) = \chi(\varepsilon\xi)$. *Then for each* $\alpha \in \mathbb{N}_0^n$ *there exist* $C(\alpha)$ *such that*

$$|\partial_\xi^\alpha \chi_\varepsilon(\xi)| \le C(\alpha)\langle\xi\rangle^{-|\alpha|}, \tag{3}$$

$$|\partial_\xi^\alpha \{\chi_\varepsilon(\xi) - 1\}| \le C(\alpha)\varepsilon\langle\xi\rangle^{1-|\alpha|}. \tag{4}$$

Proof. Since $\chi_\varepsilon^{(\alpha)}(\xi) = \varepsilon^{|\alpha|}\chi^{(\alpha)}(\varepsilon\xi)$, we have

$$|\chi_\varepsilon^{(\alpha)}(\xi)| \le C_\alpha \varepsilon^{|\alpha|}, \tag{5}$$

$$|\chi_\varepsilon^{(\alpha)}(\xi)| \le C_\alpha |\xi|^{-|\alpha|} \tag{6}$$

with $C_\alpha = \max\{\|\chi^{(\alpha)}\|_{L^\infty}, \||\xi|^{|\alpha|}\chi^{(\alpha)}(\xi)\|_{L^\infty}\}$. Then (3) follows from (5) for $|\xi| \le 1$ and from (6) for $|\xi| \ge 1$.

By the mean value theorem $|\chi_\varepsilon(\xi) - 1| = |\chi_\varepsilon(\xi) - \chi_\varepsilon(0)| \le \|\nabla\chi_\varepsilon\|_{L^\infty}|\xi| \le C\varepsilon|\xi|$. This implies (4) for $\alpha = 0$. Estimate (4) for $\alpha \ne 0$ and $|\xi| \le 1$ follows from (5). Interpolating (5) and (6), we get $|\chi_\varepsilon^{(\alpha)}(\xi)| \le C_\alpha \varepsilon |\xi|^{1-|\alpha|}$, which gives (4) for $\alpha \ne 0$ and $|\xi| \ge 1$. □

Proof of Theorem 1. Step 1. We first prove the theorem, assuming that for any $\alpha \in \mathbb{N}_0^n$ there exists $C(n, \alpha) > 0$ such that

$$|\partial_\xi^\alpha a(x,\xi)| \le C(n,\alpha)\langle\xi\rangle^{-n-1}, \qquad x \in \mathbb{R}^n, \quad \xi \in \mathbb{R}^n. \tag{7}$$

Under this assumption the integral

$$K(x,z) := (2\pi)^{-n} \int_{\mathbb{R}^n} e^{iz\xi} a(x,\xi)\, d\xi \tag{8}$$

converges for any $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^n$, and (1) holds by Fubini's theorem. Integration by parts and (7) give

$$(-iz)^\alpha K(x,z) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{iz\xi} \partial_\xi^\alpha a(x,\xi)\, d\xi, \qquad z \in \mathbb{R}^n, \quad \alpha \in \mathbb{N}_0^n.$$

Hence

$$|z|^{n+1}|K(x,z)| \le C(n) \sum_{j=1}^n |z_j|^{n+1}|K(x,z)| \le C(n)|a|_{n+1}^{(-\sigma)} \int_{\mathbb{R}^n} \langle\xi\rangle^{-n-\sigma-1}\, d\xi, \tag{9}$$

which gives (2) for $|z| \ge 1$.

In order to evaluate $K(x,z)$ for $|z| \le 1$ we choose a function $\eta \in C_0^\infty(\mathbb{R}^n)$ satisfying

$$0 \le \eta(\xi) \le 1 \text{ for } \xi \in \mathbb{R}^n, \quad \eta(\xi) = 1 \text{ for } |\xi| \le 2^{-1}, \quad \operatorname{supp}\eta \subset \{\xi \in \mathbb{R}^n : |\xi| < 1\},$$

set $\eta_R(\xi) = \eta(\xi/R)$ for $R > 1$, and write

$$K(x,z) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{iz\xi}\eta_R(\xi)a(x,\xi)\,d\xi + (2\pi)^{-n}\int_{\mathbb{R}^n} e^{iz\xi}(1-\eta_R(\xi))a(x,\xi)\,d\xi$$
$$=: J_0 + J_1.$$

Since $|\partial_\xi^\alpha \eta_R(\xi)| \le C(\alpha)\langle\xi\rangle^{-|\alpha|}$ by Lemma 2, we have

$$\left|\partial_\xi^\alpha\{(1-\eta_R(\xi))a(x,\xi)\}\right| \le \left|\partial_\xi^\alpha a(x,\xi)\right| + \sum_{\beta\le\alpha}\binom{\alpha}{\beta}\left|\partial_\xi^\beta\eta_R(\xi)\cdot\partial_\xi^{\alpha-\beta}a(x,\xi)\right|$$
$$\le C(\alpha)|a|_{n+1}^{(-\sigma)}\langle\xi\rangle^{-\sigma-|\alpha|}.$$

Hence for $|\alpha| = n$

$$|z^\alpha J_1| = \left|(2\pi)^{-n}\int_{\mathbb{R}^n} e^{iz\xi}\partial_\xi^\alpha\{(1-\eta_R(\xi)a(x,\xi))\}\,d\xi\right|$$
$$\le C\int_{|\xi|\ge R/2}|a|_{n+1}^{(-\sigma)}\langle\xi\rangle^{-n-\sigma}\,d\xi$$
$$\le C|a|_{n+1}^{(-\sigma)}R^{-\sigma},$$

which yields $|J_1| \le C|a|_{n+1}^{(-\sigma)}R^{-\sigma}|z|^{-n}$.

As for J_0 we have $|J_0| \le (2\pi)^{-n}|a|_0^{(-\sigma)}I(R)$ with $I(R) = \int_{|\xi|\le R}\langle\xi\rangle^{-\sigma}\,d\xi$, which is evaluated as follows:

(i) if $\sigma > n$, $I(R) \le \int_{\mathbb{R}^n}\langle\xi\rangle^{-\sigma}\,d\xi < \infty$;
(ii) if $\sigma = n$, $I(R) \le C(n)\int_0^R r^{n-1}\langle r\rangle^{-n}\,dr \le C(n)(1+\int_1^R r^{-1}\,dr) \le C(n)(1+\log R)$;
(iii) if $0 < \sigma < n$, $I(R) \le \int_{|\xi|\le R}|\xi|^{-\sigma}\,d\xi = C(n,\sigma)R^{n-\sigma}$.

Setting $R = |z|^{-1}$ in the estimates for J_0 and J_1, we obtain (2) for $0 < |z| \le 1$.

Step 2. For the general case we fix $\chi \in \mathcal{S}(\mathbb{R}^n)$ with $\chi(0) = 1$ and set $\chi_\varepsilon(\xi) = \chi(\varepsilon\xi)$ for $0 < \varepsilon < 1$. Since $a_\varepsilon(x,\xi) := \chi_\varepsilon(\xi)a(x,\xi)$ satisfies (7) with $C(n,\alpha)$ replaced by $C(n,\alpha,\varepsilon)$, if we set

$$K_\varepsilon(x,z) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{iz\xi}\chi_\varepsilon(\xi)a(x,\xi)\,d\xi,$$

then it follows from the result in Step 1 that $|K_\varepsilon(x,z)| \le C(n,\sigma)|a_\varepsilon|_{n+1}^{(-\sigma)}H_\sigma(z)$ and

$$a_\varepsilon(x,D)f(x) = \int_{\mathbb{R}^n} K_\varepsilon(x,x-y)f(y)\,dy, \qquad f\in\mathcal{S}(\mathbb{R}^n).$$

By Lemma 2 and the Leibniz formula, we see that $|\chi_\varepsilon(\xi)a(x,\xi)|_{n+1}^{(-\sigma)} \le C|a|_{n+1}^{(-\sigma)}$ with C independent of ε. Therefore

$$|K_\varepsilon(x,z)| \le C(n,\sigma)|a|_{n+1}^{(-\sigma)}H_\sigma(z). \tag{10}$$

Unlike Step 1 we can not define $K(x,z)$ by (8), since the integral in (8) does not necessarily converge. Instead, we choose a positive integer k with $2k+\sigma > n+1$ and define $K(x,z)$ by

$$K(x,z) = (2\pi)^{-n}|z|^{-2k} \int_{\mathbb{R}^n} e^{iz\xi}(-\Delta_\xi)^k a(x,\xi)\, d\xi$$

for $z \neq 0$, where $\Delta_\xi = \sum_{j=1}^n \partial_{\xi_j}^2$. The argument below will show that $K(x,z)$ is well defined and independent of k.

Integration by parts gives

$$|z|^{2k} K_\varepsilon(x,z) = |z|^{2k} K(x,z) + (2\pi)^{-n} \int_{\mathbb{R}^n} e^{iz\xi}(-\Delta_\xi)^k \{(\chi_\varepsilon(\xi)-1)a(x,\xi)\}\, d\xi.$$

By the Leibniz formula and Lemma 2

$$\begin{aligned} \left|(-\Delta_\xi)^k\{(1-\chi_\varepsilon(\xi))a(x,\xi)\}\right| &= \left| \sum_{|\alpha|=2k} \sum_{\beta\le\alpha} C_{\alpha\beta} \partial_\xi^\beta (\chi_\varepsilon(\xi)-1) \cdot \partial_\xi^{\alpha-\beta} a(x,\xi) \right| \\ &\le C \sum_{|\alpha|=2k} \sum_{\beta\le\alpha} \varepsilon \langle\xi\rangle^{1-|\beta|} \langle\xi\rangle^{-\sigma-|\alpha-\beta|} \\ &\le C\varepsilon \langle\xi\rangle^{1-2k-\sigma}. \end{aligned}$$

Noting $\langle\xi\rangle^{1-2k-\sigma} \in L^1(\mathbb{R}^n)$, we have

$$\lim_{\varepsilon\to+0} K_\varepsilon(x,z) = K(x,z) \quad \text{for } z \neq 0. \tag{11}$$

Since $H_\sigma(x-y)f(y)$ is integrable as a function of y for $f \in \mathcal{S}(\mathbb{R}^n)$, the dominated convergence theorem and (10) yield

$$a(x,D)f(x) = \lim_{\varepsilon\to+0} a_\varepsilon(x,D)f(x) = \lim_{\varepsilon\to+0} \int_{\mathbb{R}^n} K_\varepsilon(x,x-y)f(y)\, dy = \int_{\mathbb{R}^n} K(x,x-y)f(y)\, dy.$$

The estimate for $K(x,x-y)$ follows from (10) and (11). □

Corollary 3. *Let p, q and σ satisfy either of the following conditions:*

(1) $\sigma > 0$, $1 \le p \le q \le \infty$, $p^{-1} - q^{-1} < \sigma/n$;
(2) $0 < \sigma < n$, $1 < p < q < \infty$, $p^{-1} - q^{-1} = \sigma/n$.

Then for $a(x,\xi) \in S^{-\sigma}(\mathbb{R}^n \times \mathbb{R}^n)$ the pseudo-differential operator $a(x,D)$ is a bounded linear operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, and satisfies

$$\|a(x,D)\|_{L^p(\mathbb{R}^n)\to L^q(\mathbb{R}^n)} \le C(n,\sigma,p,q)|a|_{n+1}^{(-\sigma)}.$$

Proof. We first consider case (1). It is easy to see that $H_\sigma \in L^r(\mathbb{R}^n)$ for $1 \le r \le \infty$ if $\sigma > n$, that $H_\sigma \in L^r(\mathbb{R}^n)$ for $1 \le r < \infty$ if $\sigma = n$, and that $H_\sigma \in L^r(\mathbb{R}^n)$ for $1 \le r < \infty$

with $(\sigma - n)r > -n$, i.e. $1 - r^{-1} < \sigma/n$ if $0 < \sigma < n$. If we define $r \in [1, \infty]$ so that $p^{-1} - q^{-1} = 1 - r^{-1}$, then the corollary follows from Theorem 1 and the Young inequality that asserts $L^r(\mathbb{R}^n) * L^p(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$.

We next consider case (2). By Theorem 1

$$|a(x,D)f(x)| \leq C\{\chi_0 * |f|(x) + (\chi_1 H_\sigma) * |f|(x)\}, \quad f \in \mathcal{S}(\mathbb{R}^n),$$

where $\chi_0(x) = |x|^{\sigma - n}$ and $\chi_1(x)$ is the characteristic function of the set $\{x \in \mathbb{R}^n : |x| \geq 1\}$. Evaluating $\chi_0 * |f|$ by the Hardy-Littlewood-Sobolev inequality and $(\chi_1 H_\sigma) * |f|$ by the Young inequality, we get the corollary. □

References

1) Kumano-go H (1981) *Pseudo-differential operators.* MIT Press, Cambridge/Massachusetts/London

2) Taylor M. E (1996) *Partial differential equations II. Qualitative studies of linear equations.* Applied Mathematical Sciences, vol. 116, Springer-Verlag, New York

3) Shibata Y (2006) *Lebesgue integration.* Uchida Rokakuho, Tokyo (in Japanese)

4) Kumano-go H, Nagase M (1978) Pseudo-differential operators with non-regular symbols and applications. Funkcial. Ekvac. 21, 151–192