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Kernel estimates for pseudo-differential operators of negative order II

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In the previous paper [1] we considered a pseudo-differential operator of negative order and showed that its kernel is estimated by an L_1 function. If its symbol is related to an elliptic operator, we can conclude that the pseudo-differential operator has a kernel of exponential decay. This result was covered in [2], but an argument was omitted concerning the improper integral of a function that is not integrable. In this paper, we complete the proof given in [2].

We start by giving assumptions for the symbol of an elliptic operator. Let $m > 0$ be an even integer, and assume that

$$a(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha, \qquad x \in \mathbb{R}^n,\, \xi \in \mathbb{R}^n$$

satisfies

$$|a_\alpha(x)| \le M_A, \tag{1}$$

$$|a(x,\xi)| \ge \delta_A |\xi|^m, \tag{2}$$

$$|\arg a(x,\xi)| \le \kappa_A \tag{3}$$

for any $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$ with some constants $M_A > 0$, $\delta_A \in (0,1]$ and $\kappa_A \in [0,\pi)$. By $C = C(A)$ we mean that a constant C depends only on n, m, M_A, δ_A and κ_A. We note that a strongly elliptic operator satisfies conditions (1)-(3). In other words, the inequalities

$$|a_\alpha(x)| \le M_A, \qquad \operatorname{Re} a(x,\xi) \ge \delta_A |\xi|^m$$

for $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$ imply (2) and (3) with $\kappa_A = \arctan\big(\delta_A^{-1} M_A \sum_{|\alpha|=m} 1\big)$. We set

$$\begin{aligned}
\Lambda_A &= \{\lambda \in \mathbb{C} :\, \kappa_A < \arg\lambda < 2\pi - \kappa_A,\, \lambda \ne 0\},\\
\Sigma_A &= \{\zeta \in \mathbb{C} :\, |\arg\zeta| \le \kappa_A,\, \zeta \ne 0\},\\
d(\lambda) &= \operatorname{dist}(\lambda, \Sigma_A).
\end{aligned}$$

When we construct a parametrix for the elliptic operator $a(x,D) - \lambda$ with $\lambda \in \Lambda_A$, the pseudo-differential operators with symbols of the form

$$p(x,\xi,\lambda) = \frac{\xi^\beta}{(a(x,\xi)-\lambda)^k} \tag{4}$$

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appear, where $k \in \mathbb{N}$ and $\beta \in \mathbb{N}_0^n$ satisfy $mk > \beta$. Here $\lambda \in \Lambda_A$ is regarded as a parameter. If we set

$$\sigma = mk - \beta, \tag{5}$$

then $p(x, \xi, \lambda) \in S^{-\sigma}(\mathbb{R}_x^n \times \mathbb{R}_\xi^n)$, the symbol class of order $-\sigma$ and type $(1, 0)$.

For $\rho \in \mathbb{R}$, $c \geq 0$ and $s > 0$ we set

$$\Phi_\rho(s, c) = e^{-cs} \times \begin{cases} 1 & (\rho > 0), \\ 1 + \log_+ s^{-1} & (\rho = 0), \\ s^\rho & (\rho < 0), \end{cases} \qquad q_\rho^n(s) = \int_0^\infty \frac{r^{n-2} e^{-sr}}{(r+1)^{\rho-1}}\, dr, \tag{6}$$

where $\log_+ t = \max\{1, t\}$ for $t > 0$.

Theorem 1. *Let $p(x, \xi, \lambda)$ and σ be as in (4), (5), and assume $\sigma \geq 2$. Then the pseudo-differential operator $p(x, D, \lambda)$ is written as*

$$p(x, D, \lambda) f(x) = \int_{\mathbb{R}^n} K_\lambda(x, x-y) f(y)\, dy,$$

where the function $K_\lambda(x, x-y)$ satisfies

$$|K_\lambda(x, x-y)| \leq C \left(\frac{|\lambda|}{d(\lambda)} \right)^k |\lambda|^{(n-\sigma)/m} \Phi_{\sigma-n}(d(\lambda)|\lambda|^{-1+1/m}|x-y|, c_0)$$

with some constants $C = C(A, k, \beta) > 0$ and $c_0 = c_0(A) > 0$.

Remark. *Theorem 1 is also valid for $\sigma = 1$. However, this case is very subtle, since the integrand that appears in Lemma 4 below is not integrable. We will discuss this case in a forthcoming paper.*

For the proof of Theorem 1 we prepare several lemmas as follows.

Lemma 2. *For $\zeta \in \Sigma_A$ and $\lambda \in \Lambda_A$,*

$$|\zeta - \lambda| \geq \frac{d(\lambda)}{4|\lambda|}(|\zeta| + |\lambda|).$$

Proof. If $|\zeta| \geq 2|\lambda|$, then $|\zeta - \lambda| \geq |\zeta| - |\lambda| \geq \frac{1}{2}|\zeta| \geq \frac{1}{4}(|\zeta| + |\lambda|)$. If $|\zeta| \leq 2|\lambda|$, then $|\zeta - \lambda| \geq d(\lambda) = \frac{d(\lambda)}{3|\lambda|}(2|\lambda| + |\lambda|) \geq \frac{d(\lambda)}{3|\lambda|}(|\zeta| + |\lambda|)$. The lemma follows from these inequalities and $d(\lambda) \leq |\lambda|$. □

Lemma 3. *There exist $C_1 = C_1(n, m, M_A, \delta_A) > 0$ such that if $\zeta \in \mathbb{C}^n$ and $\xi \in \mathbb{R}^n$ satisfy $|\zeta| \leq C_1 \frac{d(\lambda)}{|\lambda|}(|\xi| + |\lambda|^{1/m})$ then*

$$|a(x, \xi + \zeta) - \lambda| \geq \frac{d(\lambda)}{8|\lambda|}(\delta_A |\xi|^m + |\lambda|)$$

for $\lambda \in \Lambda_A$.

Proof. Applying the inequality $ab \le a^p/p + b^q/q$ for $a \ge 0$, $b \ge 0$, $p > 0$, $q > 0$ with $1/p + 1/q = 1$, we have, for any $\varepsilon \in (0,1)$,

$$\begin{aligned}|a(\xi+\zeta) - a(\xi)| &\le C \sum_{j=0}^{m-1} |\xi|^j |\zeta|^{m-j} \\ &\le C|\zeta|^m + C\sum_{j=1}^{m-1} (\varepsilon^{1/m}|\xi|)^j (\varepsilon^{-j/m(m-j)}|\zeta|)^{m-j} \\ &\le C|\zeta|^m + \sum_{j=1}^{m-1}\left(\frac{j}{m}\varepsilon|\xi|^m + \frac{m-j}{m}C^{m/(m-j)}\varepsilon^{-j/(m-j)}|\zeta|^m\right) \\ &\le \frac{m-1}{2}\varepsilon|\xi|^m + C'\varepsilon^{1-m}|\zeta|^m.\end{aligned}$$

Hence Lemma 2 gives

$$\begin{aligned}|a(\xi+\zeta) - \lambda| &\ge |a(\xi) - \lambda| - |a(\xi+\zeta) - a(\xi)| \\ &\ge \frac{d(\lambda)}{4|\lambda|}(\delta_A|\xi|^m + |\lambda|) - m\varepsilon|\xi|^m - C'\varepsilon^{1-m}|\zeta|^m.\end{aligned}$$

If we take ε and ζ so that

$$m\varepsilon = \frac{d(\lambda)}{16|\lambda|}\delta_A, \qquad C'\varepsilon^{1-m}|\zeta|^m \le \frac{d(\lambda)}{16|\lambda|}(\delta_A|\xi|^m + |\lambda|),$$

then we get, with $\varepsilon \in (0,1)$,

$$|a(\xi+\zeta) - \lambda| \ge \frac{d(\lambda)}{8|\lambda|}(\delta_A|\xi|^m + |\lambda|)$$

for $|\zeta|^m \le \frac{1}{16C'}(\frac{\delta_A}{16m})^{m-1}(\frac{d(\lambda)}{|\lambda|})^m(\delta_A|\xi|^m + |\lambda|)$. From this inequality the lemma follows. □

Lemma 4. *Let $p(x,\xi,\lambda)$ and σ be as in (4), (5), and assume $\sigma \ge 2$. For $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^n \setminus \{0\}$ we set $\omega_z = z/|z|$ and $\Pi_z = \{\eta \in \mathbb{R}^n : \eta \perp z\}$, and define $F_{x,z}(u,\eta,\lambda)$ by*

$$F_{x,z}(u,\eta,\lambda) = (2\pi)^{-1}\int_{\mathbb{R}} e^{iut} p(x, t\omega_z + \eta, \lambda)\, dt, \qquad u \in \mathbb{R},\ \eta \in \Pi_z,\ \lambda \in \Lambda_A.$$

Then there exist $C = C(A,k,\beta) > 0$ and $c_0 = c_0(A) > 0$ such that

$$|F_{x,z}(u,\eta,\lambda)| \le C\left(\frac{|\lambda|}{d(\lambda)}\right)^k (|\eta| + |\lambda|^{1/m})^{1-\sigma} \exp\left(-c_0\frac{d(\lambda)}{|\lambda|}(|\eta| + |\lambda|^{1/m})|u|\right).$$

Proof. Set $\tau_0 = C_1\frac{d(\lambda)}{2|\lambda|}(|t| + |\eta| + |\lambda|^{1/m})$ for $t \in \mathbb{R}$, $\eta \in \Pi_z$, $\lambda \in \Lambda_A$, where C_1 is the constant in Lemma 3. Let $u \ge 0$. Then by Lemma 3 we can deform the integral path from $\mathbb{R}$ to $\mathbb{R} + i\tau_0$

to get

$$F_{x,z}(u,\eta,\lambda) = (2\pi)^{-1}\int_{\mathbb{R}} e^{iut}\frac{(t\omega_z+\eta)^{\beta}}{(a(x,t\omega_z+\eta)-\lambda)^k}\,dt$$
$$= (2\pi)^{-1}\int_{\mathbb{R}} e^{iut-u\tau_0}\frac{((t+\mathrm{i}\tau_0)\omega_z+\eta)^{\beta}}{(a(x,(t+\mathrm{i}\tau_0)\omega_z+\eta)-\lambda)^k}\,dt.$$

Lemma 3 and the change of variables $t = (|\eta|+|\lambda|^{1/m})s$ give

$$|F_{x,z}(u,\eta,\lambda)| \le C\int_{\mathbb{R}} e^{-u\tau_0}\left(\frac{|\lambda|}{d(\lambda)}\right)^k (|t|+|\eta|+|\lambda|^{1/m})^{|\beta|-mk}\,dt$$
$$\le C\exp\left(-\frac{C_1 d(\lambda)}{2|\lambda|}(|\eta|+|\lambda|^{1/m})u\right)\left(\frac{|\lambda|}{d(\lambda)}\right)^k (|\eta|+|\lambda|^{1/m})^{1-\sigma},$$

since $0\le\tau\le\tau_0$ implies $|\mathrm{i}\tau| \le C_1\frac{d(\lambda)}{|\lambda|}(|t\omega_z+\eta|+|\lambda|^{1/m})$. Here we also used the fact that the integrand is integrable for $\sigma\ge 2$.

The case $u<0$ can be treated similarly if we deform the integral path from $\mathbb{R}$ to $\mathbb{R}-\mathrm{i}\tau_0$. □

Lemma 5. *Let Φ_ρ and q^n_ρ be as in (6). Then for $\sigma>0$ there exists $C=C(n,\sigma)$ such that*

$$0 < q^n_\sigma(s) \le C\Phi_{\sigma-n}(s,0), \qquad s>0.$$

Proof. It is clear that $q^n_\sigma(s)>0$. So we need only show $q^n_\sigma(s)\le C\Phi_{\sigma-n}(s,0)$. If $\sigma>n$, then

$$q^n_\sigma(s) \le \int_0^\infty \frac{r^{n-2}}{(r+1)^{\sigma-1}}\,dr < \infty,$$

since the integrand is $O(r^{n-2})$ as $r\to+0$ and $O(r^{n-\sigma-1})$ as $r\to\infty$.

If $\sigma=n$, then

$$q^n_\sigma(s) \le \int_0^\infty \frac{e^{-sr}}{r+1}\,dr \le \int_0^{1/s}\frac{1}{r+1}\,dr + \int_{1/s}^\infty e^{-sr}\frac{dr}{r} = \log(1+s^{-1}) + \int_1^\infty e^{-r}\frac{dr}{r}.$$

If $0<\sigma<n$, then

$$q^n_\sigma(s) \le \int_0^\infty r^{n-\sigma-1}e^{-sr}\,dr = s^{\sigma-n}\int_0^\infty r^{n-\sigma-1}e^{-r}\,dr.$$

Thus the lemma follows from the above inequalities. □

Proof of Theorem 1. Step 1. As was shown in the proof of [1, Theorem 1], the kernel of $p(x,\xi,\lambda)$ is given by $K_\lambda(x,x-y)$, where the function $K_\lambda(x,z)$ is formally written as

$$K_\lambda(x,z) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{\mathrm{i}z\xi}p(x,\xi,\lambda)\,d\xi, \qquad x\in\mathbb{R}^n,\ z\in\mathbb{R}^n. \tag{7}$$

The integral in (7) should be interpreted as an oscillatory integral, namely,

$$K_\lambda(x,z)=\lim_{\varepsilon\to+0}K_{\lambda,\varepsilon}(x,z),\qquad K_{\lambda,\varepsilon}(x,z)=(2\pi)^{-n}\int_{\mathbb{R}^n}e^{\mathrm{i}z\xi}p(x,\xi,\lambda)\chi_\varepsilon(\xi)\,d\xi \tag{8}$$

for $z\neq 0$, where $\chi_\varepsilon(\xi)=\chi(\varepsilon\xi)$ for arbitrary $\chi\in\mathcal{S}(\mathbb{R}^n)$ with $\chi(0)=1$. This limit is independent of the choice of χ.

We may assume that $|\lambda|=1$, since the change of variables $\xi=|\lambda|^{1/m}\eta$ and the relation $p(x,|\lambda|^{1/m}\eta,\lambda)=|\lambda|^{-\sigma/m}p(x,\eta,\lambda/|\lambda|)$ give

$$K_\lambda(x,z)=(2\pi)^{-n}\int_{\mathbb{R}^n}e^{\mathrm{i}|\lambda|^{1/m}z\eta}p(x,|\lambda|^{1/m}\eta,\lambda)|\lambda|^{n/m}\,d\eta=|\lambda|^{(n-\sigma)/m}K_{\lambda/|\lambda|}(x,|\lambda|^{1/m}z).$$

In addition, we give the proof only for $n\geq 2$, since the case $n=1$ can be treated with a slight modification.

Step 2. Let $|\lambda|=1$. We proceed by a formal calculation, which will be justified in Step 3. We note that the justification is not needed for $\sigma>n$ since $p(x,\xi,\lambda)$ is integrable in ξ. Let ω_z, Π_z and $F_{x,z}(u,\eta,\lambda)$ be as in Lemma 4. In (7) we change the variables $\xi=t\omega_z+\eta$ with $t\in\mathbb{R}$, $\eta\in\Pi_z$ to have

$$K_\lambda(x,z)=(2\pi)^{-n}\int_{\Pi_z}d\eta\int_{\mathbb{R}}e^{\mathrm{i}|z|t}p(x,t\omega_z+\eta,\lambda)\,dt=(2\pi)^{1-n}\int_{\Pi_z}F_{x,z}(|z|,\eta,\lambda)\,d\eta.$$

Using Lemma 4 and the polar coordinates, we get

$$\begin{aligned}|K_\lambda(x,z)|&\leq Cd(\lambda)^{-k}\int_{\Pi_z}(|\eta|+1)^{1-\sigma}\exp(-c_0d(\lambda)(|\eta|+1)|z|)\,d\eta\\&\leq Cd(\lambda)^{-k}\int_0^\infty\frac{r^{n-2}}{(r+1)^{\sigma-1}}\exp(-c_0d(\lambda)(r+1)|z|)\,dr\\&=Cd(\lambda)^{-k}\exp(-c_0d(\lambda)|z|)q_\sigma^n(c_0d(\lambda)|z|).\end{aligned}$$

Thus the desired result follows from Lemma 5 and $q_\sigma^n(c_0s)\leq C(c_0,\sigma)q_\sigma^n(s)$.

Step 3. Let $|\lambda|=1$. We justify the calculation in Step 2. To this end we set $\chi^l(\eta)=\exp(-|\eta|^2)$ for $\eta\in\mathbb{R}^l$, $l\in\mathbb{N}$, and $\chi_\varepsilon^l(\eta)=\chi^l(\varepsilon\eta)$ for $\varepsilon>0$. As the function $\chi_\varepsilon(\xi)$ in (8) we choose $\chi_\varepsilon^n(\xi)$, and use the relation $\chi_\varepsilon^n(\xi)=\chi_\varepsilon^1(t)\chi_\varepsilon^{n-1}(\eta)$ for $\xi=t\omega_z+\eta$ with $t\in\mathbb{R}$, $\eta\in\Pi_z$. We also use the identity

$$\mathcal{F}^{-1}(\mathcal{F}f\cdot\mathcal{F}g)=f*g$$

for $f,g\in\mathcal{S}(\mathbb{R})$, and hence for $f\in L_2(\mathbb{R})$, $g\in\mathcal{S}(\mathbb{R})$, where $\mathcal{F}f$ stands for the Fourier transform of f. Then we have

$$\begin{aligned}K_{\lambda,\varepsilon}(x,z)&=(2\pi)^{-n}\int_{\Pi_z}d\eta\int_{\mathbb{R}}e^{\mathrm{i}|z|t}p(x,t\omega_z+\eta,\lambda)\chi_\varepsilon^1(t)\chi_\varepsilon^{n-1}(\eta)\,dt\\&=(2\pi)^{1-n}\int_{\Pi_z}\chi_\varepsilon^{n-1}(\eta)\,d\eta\int_{\mathbb{R}}F_{x,z}(|z|-\varepsilon u,\eta,\lambda)\mathcal{F}^{-1}\chi^1(u)\,du.\end{aligned}$$

In order to apply the dominated convergence theorem we wish to evaluate the integrand in (u,η) by an integrable function independent of ε. Lemma 4 gives

$$\begin{aligned}&\left|\chi_\varepsilon^{n-1}(\eta)F_{x,z}(|z|-\varepsilon u,\eta,\lambda)\mathcal{F}^{-1}\chi^1(u)\right|\\ &\le C\frac{d(\lambda)^{-k}}{(|\eta|+1)^{\sigma-1}}\exp\left(-\varepsilon^2|\eta|^2-c_0d(\lambda)||z|-\varepsilon u|(|\eta|+1)-4^{-1}u^2\right).\end{aligned}$$

For any $\varepsilon\in(0,1)$ and any c' with $0<2c'\le c_0$ the exponent of the above exponential function is bounded by

$$\begin{aligned}&-2c'd(\lambda)|z|(|\eta|+1)+2c'd(\lambda)\varepsilon|u|(|\eta|+1)-\varepsilon^2|\eta|^2-4^{-1}u^2\\ &\le -2c'd(\lambda)|z|(|\eta|+1)+2c'd(\lambda)|u|+2c'd(\lambda)\varepsilon|u||\eta|-\varepsilon^2|\eta|^2-4^{-1}u^2\\ &\le -2c'd(\lambda)|z|(|\eta|+1)+2c'd(\lambda)|u|+(c'd(\lambda)-4^{-1})u^2+(c'd(\lambda)-1)\varepsilon^2|\eta|^2.\end{aligned}$$

If we take $c'>0$ so that $c'd(\lambda)<8^{-1}$ and $2c'<c_0$, the exponent is bounded by

$$\begin{aligned}&-2c'd(\lambda)|z|(|\eta|+1)+2c'd(\lambda)|u|-8^{-1}(u^2+\varepsilon^2|\eta|^2)\\ &\le -2c'd(\lambda)|z|(|\eta|+1)+2c'd(\lambda)|u|-8^{-1}u^2.\end{aligned}$$

Hence the dominated convergence theorem and the observation that $F_{x,z}(u,\eta,\lambda)$ is continuous in u yield

$$\begin{aligned}K_\lambda(x,z)&=(2\pi)^{1-n}\int_{\Pi_z}\chi^{n-1}(0)\,d\eta\int_{\mathbb{R}}F_{x,z}(|z|,\eta,\lambda)\mathcal{F}^{-1}\chi^1(u)\,du\\ &=(2\pi)^{1-n}\int_{\Pi_z}F_{x,z}(|z|,\eta,\lambda)\,d\eta.\end{aligned}$$

Thus the calculation in Step 2 has been justified. □

References

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2) Shimakura N (1992) *Partial differential operators of elliptic type.* Translations of Mathematical Monographs 99, AMS, Providence